Kinematics of A Flexible Link

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1 Introduction

The kinematics of flexible beam is a fundamental part in modeling flexible manipulators and has raised great interest in the past twenty years. To describe link deformation, most of researches took the link's deflections as the deformation coordinates [1, 2]. This approach has the advantages of intuitive modeling procedure since it is coupled with the use of natural coordinates. However, the problem raises when we perform the real-time simulation and control since it is difficult to detect the link deflection using conventional measurement systems.

Rather than link deflections, one can use the local curvatures as the deformation coordinates [3]. The link deformation in this approach is described in terms of local curvatures and related to a floating frame attached to the flexible link. This approach yields a small number of equations and is more effective for real-time simulation and control purposes since it is easily interfaced with on-line strain measurements [4].

In this paper, we develop a kinematic model of a flexible beam using local curvatures as the deformation coordinates. First we define the local frames and make basic assumptions. Then we consider the orientation and position variations of the local frame. At the end we give the endpoint position and orientation in terms of local curvatures.

2 Flexible-Link Kinematics

The kinematic model of a manipulator implies finding the relationships between the various reference frames attached to the manipulator. These relationships are completely specified by the rotation matrix (or vector) and position vector. The kinematic model of a flexible link is constructed similarly to the robotic manipulator. To describe position and orientation of the flexible link, we divide the flexible link into n sections. Each section has the same length Δs in its neutral axis. A local frame is assigned to each section and represents the position and orientation of the section. The origin of the local frame is located at the left point of the neutral axis of the section with its x axis tangent to the neutral axis. All the local frames have the same orientation when the link undergoes no deformation. In order to represent the endpoint, two frames are assigned to the last section, one has its origin at the left point of neutral axis and another has its origin at right point of the neutral axis.

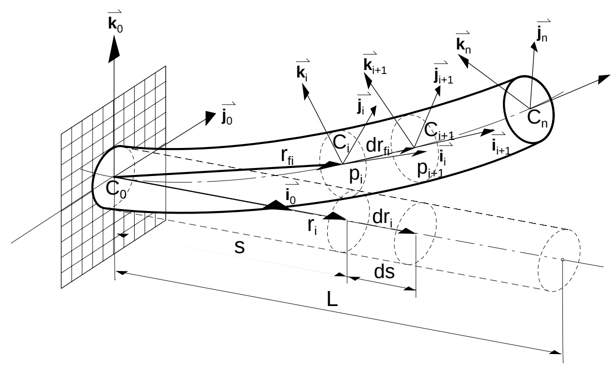

Figure 1: Flexible Link and Its Frames

Fig. 1 shows the flexible beam with the four reference frames that are of interest. Specifically, we want to derive the relation between the base frame $\{C_0\}$ and the endpoint frame $\{C_n\}$. To build the model, we also make the following assumptions:

1. The link is considered as an Euler-Bernoulli beam, implying that the beam sections stay in plane and perpendicular to the neutral axis.
2. The deformations are kept within the elastic limit of the beam material and there is no permanent deformation.
3. The neutral axis is non-extensible. Hence, the longitudinal deformation is ignored.
4. The link has a circular cross-section so no wrapping occurs.

To derive the model, we first consider the orientation variations of the frames $\{C_i\}$. We assume that the time is fixed and consider only the variation about space variable. Realizing that as $n \to \infty$, the arc length $\Delta s \rightarrow ds$, the orientation variations of the frame $\{C_i\}$ can be expressed as

$$ {}^{i}\delta\boldsymbol{\theta} = \frac{{}^{i}\partial\boldsymbol{\theta}}{\partial s}ds = {}^{i}\boldsymbol{\kappa}ds \tag{1} $$

where the leading superscript i indicates that the quantity is defined in frame $\{C_i\}$ and ${}^{i}\boldsymbol{\kappa}$ represents the local curvature vector and is defined by

$$ {}^{i}\boldsymbol{\kappa} = \lim_{s\to 0}\frac{\Delta\boldsymbol{\theta}}{\Delta s} \Longleftrightarrow \begin{bmatrix} {}^{i}\kappa_x \\ {}^{i}\kappa_y \\ {}^{i}\kappa_z \end{bmatrix} = \begin{bmatrix} \frac{{}^{i}\partial\theta_x}{\partial s} \\ \frac{{}^{i}\partial\theta_y}{\partial s} \\ \frac{{}^{i}\partial\theta_z}{\partial s} \end{bmatrix} \tag{2} $$

Eq. 1 gives the orientation variation defined in frame $\{C_i\}$. In many cases, we need to express the orientation in the base frame $\{C_0\}$. This can be obtained through the mapping of the frame $\{C_i\}$ to the frame $\{C_0\}$ and the orientation variation in frame $\{C_0\}$ is given by

$$ {}^{0}\delta\boldsymbol{\theta} = {}^{0}_{i}R\,{}^{i}\boldsymbol{\kappa}ds \tag{3} $$

where ${}^{0}_{i}R$ represents the rotation matrix of the frame $\{C_i\}$ relative to base frame $\{C_0\}$.

The same kind of relation exists between the rotation matrix ${}^{0}_{i}R$ and ${}^{i}\boldsymbol{\kappa}$ as between the rotation matrix and angular velocity vector. This relation leads to the following differential equation [3]:

$$ \frac{d\,{}^{0}_{i}R}{ds} = {}^{0}_{i}R\,{}^{i}\tilde{\boldsymbol{\kappa}} \tag{4} $$

The tilde symbol in ${}^{i}\tilde{\boldsymbol{\kappa}}$ indicates that ${}^{i}\tilde{\boldsymbol{\kappa}}$ is the skew-symmetric matrix formed with the elements of ${}^{i}\boldsymbol{\kappa}$.

The rotation matrix can now be found by solving the differential of Eq. 4. Piedbœuf [3] has proposed a simple approach to solve this differential equation by separating the rotation matrix into different order terms. Now we directly use the result presented by Piedbœuf. To simplify notation, the following definitions are used:

$$ v = \int_0^s\int_0^\xi {}^{i}\kappa_z d\eta d\xi \quad w = -\int_0^s\int_0^\xi {}^{i}\kappa_y d\eta d\xi \quad \alpha = \int_0^s {}^{i}\kappa_x d\xi \tag{5} $$

The rotation matrix from frame $\{C_i\}$ to $\{C_0\}$ is

$$ {}^{i}_{0}R = \begin{bmatrix} 1-\frac{1}{2}(v'^2+w'^2) & -v'-\int_0^s \alpha' w' d\xi & -w'+\int_0^s \alpha' v' d\xi \\ v'-\int_0^s \alpha w'' d\xi & 1-\frac{1}{2}(v'^2+\alpha^2) & -\alpha-\int_0^s v' w'' d\xi \\ w'+\int_0^s \alpha v'' d\xi & \alpha-\int_0^s v'' w' d\xi & 1-\frac{1}{2}(w'^2+\alpha^2) \end{bmatrix} \tag{6} $$

The rotation angles of $\{C_i\}$ relative to the base of the link can now be determined by integrating Eq. 3 and the resulting rotation vector is expressed as

$$ \boldsymbol{\theta} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} \alpha - \int_0^s w'v''d\xi + \int_0^s v'w''d\xi \\ -w' + \int_0^s v'\alpha' d\xi - \int_0^s \alpha v'' d\xi \\ v' - \int_0^s \alpha w'' d\xi + \int_0^s w'\alpha' d\xi \end{bmatrix} \tag{7} $$

To obtain the position vector relative to the base of the link, we consider the position variation in Fig. 1 and have

$$ {}^{0}\delta\mathbf{r}_{fi} = {}^{0}_{i}R\,{}^{i}d\mathbf{r}_{fi} = {}^{0}_{i}R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ds \tag{8} $$

Using the above equation and substituting the rotation matrix given in Eq. 6, the endpoint position vector is obtained by integration of the variation ${}^{0}\delta\mathbf{r}_{fi}$

$$ \mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} s - \frac{1}{2}\int_0^s (v'^2 + w'^2)d\xi \\ v - \int_0^s\int_0^\xi w''\alpha d\eta d\xi \\ w + \int_0^s\int_0^\xi v''\alpha d\eta d\xi \end{bmatrix} \tag{9} $$

where s is the arc length of the neutral axis from frame $\{C_0\}$ to frame $\{C_i\}$.

Eq. 7 and Eq. 9 give the orientation and position of the frame $\{C_i\}$. The orientation and position of the endpoint of the flexible link can be obtained by simply replacing the s with L, the length of the flexible link in Eq. 7 and Eq. 9 respectively.

3 Conclusion

We have developed the kinematic model of a flexible link that uses the local curvatures as the deformation coordinates. This model is more suitable for the real-time simulations since the local curvatures can be obtained using the on-line strain measurements. As long as the curvatures are detected the endpoint position and orientation can be easily determined using the forward kinematics, which is similar to the rigid manipulator.

References

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