A GENERALIZATION OF THE PRESSURE ANGLE

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ABSTRACT

The concept of pressure angle has been used for almost two hundred years. It is a key performance indicator in higher-pair mechanisms, whereby motion and force are transmitted by point or line contact. However, the pressure angle may be misleading in some specific applications because of its limited applicability range, which has been overlooked. Moreover, a few other equivalent indices are used in some specific cases, which adds to the terminology unnecessary complexity. In an attempt to unify and generalize the concepts around the issue, this paper introduces a new concept, the pressure ratio, which is based on the essential nature of the pressure angle and is applicable to all planar serial transmission trains.

Une généralisation de l'angle de pression

Le concept d'angle de pression est utilisé depuis presque deux cents ans. C'est un indicateur principal de performance dans les mécanismes à couples cinématiques supérieurs, dans lesquels le mouvement et la force sont transmis par le point ou la ligne de contact. Cependant, l'angle de pression peut mener à de mauvais résultats dans quelques applications spécifiques à cause de sa plage de validité limitée, qui a été négligée. De plus, d'autres indicateurs équivalents sont utilisés dans quelques cas spécifiques, ce qui ajoute à la terminologie une complexité inutile. Dans une tentative pour unifier et généraliser les concepts autour du problème, cet article introduit un nouveau concept, le rapport de pression, qui est basé sur la nature essentielle de l'angle de pression et est applicable à tous trains planaires de transmission en série.

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1 Introduction

Since its inception by J. V. Poncelet in 1826 [1], the pressure angle has served as an indicator of the quality of force transmission for a number of mechanisms. Another indicator, the transmission angle, was introduced by Alt [2] and developed by Hain [3]; then generalized by Sutherland and Roth [4] for the same purpose. The transmission angle targets linkage synthesis. Furthermore, Gupta and Kazerounian defined the deviation angle [5], which is the counterpart of the transmission angle and bears almost the same properties as the pressure angle, except that it is applied to linkages. Since these indices are intended for the same purpose, it seems that an integration thereof is necessary.

2 Motivation

The pressure angle in *gear transmissions* is defined as the angle between the line of action and the common tangent to the two pitch circles at the pitch point [6]. The counterpart definition in *cam transmissions* is the angle between the direction of the unit normal to the driving element pitch curve and the direction of the velocity of the contact point, as pertaining to the driven element [7, 8].

Regardless of the difference between the two foregoing definitions, the pressure angle is essentially the same: it is a quantity characterizing the force exerted by the driving element onto the driven element [9], and hence, the smaller the pressure angle, the smaller the driving or contact force.

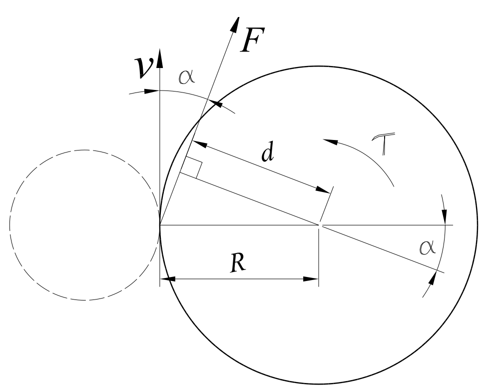


Figure 1: The pressure angle in the driving of a gear by a pinion

As shown in Fig. 1 , the pressure angle in a simple gear train is α. The corresponding contact force is given by

$$F = \frac{\tau}{d} = \frac{\tau}{R\cos\alpha} \tag{1}$$

The ratio between load τ and the pitch circle radius R of the driven gear should be constant, so that the contact force under study is independent on the size of the element.

Then, the only variable in eq. (1) is α: the bigger α, the bigger the contact force, and hence, the worse the force transmission. This shows the essence of the pressure angle.

We came across a problem when applying the pressure angle to the case where a cam is driven by its follower. Let us consider two configurations of the same mechanism, as shown in Fig. 2.

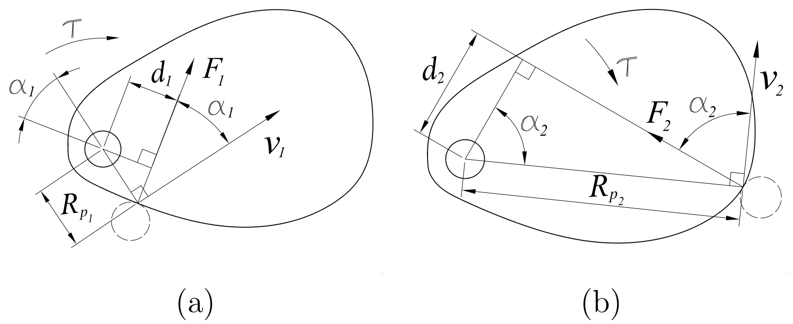


Figure 2: Two configurations of a driven cam

A static force analysis yields

$$F_1 = \frac{\tau}{d_1} = \frac{\tau}{R_{p_1}\cos\alpha_1}$$

$$F_2 = \frac{\tau}{d_2} = \frac{\tau}{R_{p_2}\cos\alpha_2}$$

Although α_1 is smaller than α_2, the increment from R_{p_1} to R_{p_2} makes F_2 smaller than F_1, which means the force transmission in configuration (b) is better than in (a) in spite of the higher pressure angle of (b). This result conflicts with the purpose of the pressure angle.

3 The Pressure Ratio

In order to provide a precise index for the force transmission quality for a mechanism like that of Fig. 2, we introduce a new concept, the *pressure ratio*:

$$p = \frac{d}{R_c} \tag{2}$$

where, d is the pressure radius, defined as the length of the lever arm of the contact force, R_c being the *characteristic radius*, which is a length representative of the size of the driven element.

We define R_c as the root mean square (rms) value of the radial coordinate $\rho(\theta)$ of the *action profile* of the driven element throughout the whole working cycle, for $0 \le \theta \le 2\pi$. Hence, the *characteristic radius* is given by

$$R_c = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\rho^2(\theta)d\theta}$$

From a geometric viewpoint, we notice that the *characteristic radius* is just the radius of a circle with the same area as the *action profile.* This circle is henceforth called the *equivalent circle.*

In the case of gears, the *action profile* is simply the pitch profile. For example, the *action profile* of a circular gear is its pitch circle. In the case of planar cams, the *action profile* is just the cam profile. However, if rollers are used in the mechanism, the cam profile must be replaced by its pitch profile, as shown in Fig. 3.

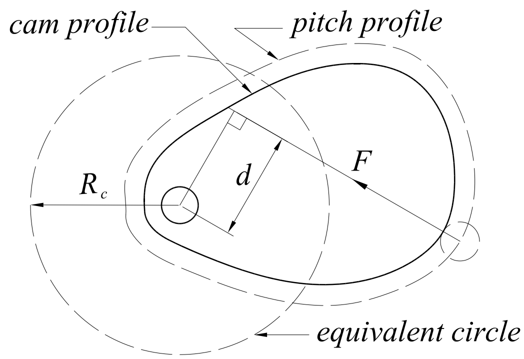


Figure 3: The action profile of a driven cam

In the case of linkages, under static, conservative conditions, the *action profile* is the circle described by the moving joint-centre of the output link, of length l_n. Hence, the *characteristic radius*, in this case, is simply l_n.

Notice that d in eq. (2) is the essential variable to determine the contact force, the *characteristic radius* making the *pressure ratio* dimensionless and invariant to scaling.

According to its definition, the pressure ratio in gear transmissions, as shown in Fig. 1, is given by

$$p = \frac{d}{R_c} = \frac{d}{R} = \cos\alpha \tag{3}$$

which is the cosine of the pressure angle.

As long as the *action profile* of the driven element is circular, such as a circular gear or a circular cam-follower, the relationship of eq. (3) between the pressure ratio and the pressure angle exists. Therefore, the pressure ratio coincides with the pressure angle in this case: the bigger the pressure ratio, the smaller the pressure angle, and the smaller the contact force in the transmission. As a result, *the pressure angle is applicable only if the action profile of the driven element is circular.*

4 Application of the Pressure Ratio

For a driven element with a non-circular action profile, the pressure ratio does not coincide with the pressure angle any longer, because the pressure angle cannot indicate the quality of force transmission, as shown in Fig. 2

We illustrate below the evaluation of the pressure ratio in such cases.

4.1 The Case of a Driven Cam

As shown in Fig. 3, the pressure ratio of the driven cam is given by

$$p = \frac{d}{R_c}$$

where, R_c is the radius of the equivalent circle, whose area is the same as that of the pitch curve.

4.2 The Case of Non-Circular Gears

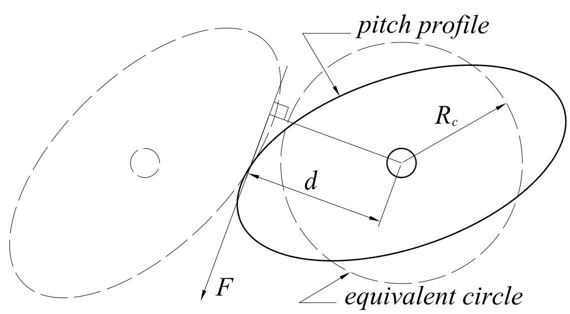


Figure 4: The pressure ratio of an elliptic-gear train

We use the elliptic-gear train as an example to illustrate the pressure ratio. As shown in Fig. 4, the pressure ratio is given by

$$p = \frac{d}{R_c}$$

Notice that *the pressure ratio* can be greater than unity, while the cosine of the pressure angle is bounded within unity. Since the computation of both the pressure ratio and the pressure angle need the direction of the driving force, *the two indices are only applicable to the mechanisms with the directions of the driving forces being determined.*

5 The Total Pressure Ratio

The pressure ratio between a pair of elements has been discussed, which can be called *the pair pressure ratio.* We investigate here *the total pressure ratio* of a planar serial train, which is the typical form of motion and force transmission, as shown in Fig. 5.

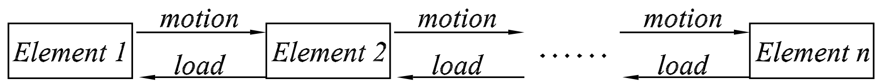


Figure 5: The motion and force transmission of a planar serial train

The motion is transferred from the first element, or the driving element, to the end or driven element, while the load is transferred inversely, from the latter to the former. Since the pressure ratio is inversely proportional to the contact force, we readily obtain, under static, conservative conditions,

$$p_1 F_1 = p_2 F_2 = ... = p_n F_n = \frac{\tau}{R_{c_n}} = \text{constant}$$

Hence, we calculate the pressure ratio p_n of the final driven element, first. Then, from the static force analysis of the previous element, we obtain the ratio between two contact forces, F_n and F_{n-1}, which yields the pressure ratio of the previous element:

$$p_{n-1} = \frac{F_n}{F_{n-1}} p_n$$

Repeating the above procedure, we can compute the pressure ratio between every pair of elements. Notice that the smallest pressure ratio will be chosen as the total pressure ratio of the system, because this reflects the biggest contact force produced during transmission.

5.1 A Gear Train

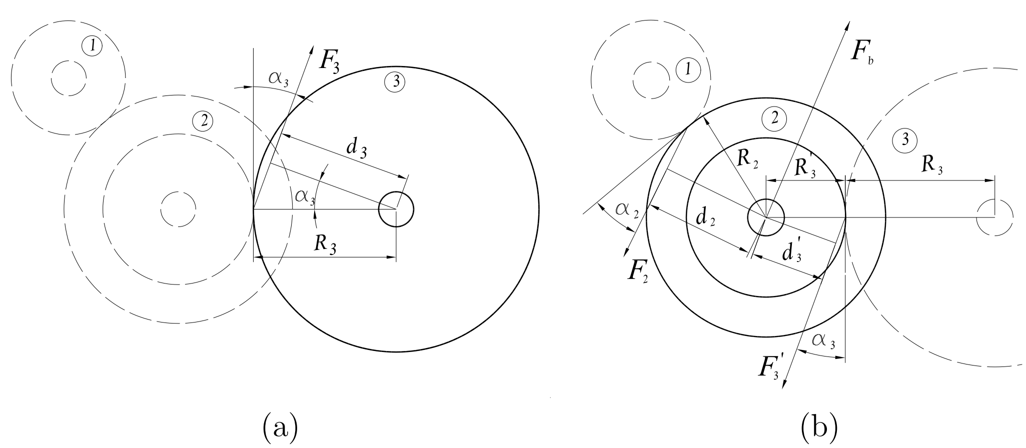


Figure 6: The free-body diagram (FBD) of (a) the driven gear, and (b) the middle gear

Shown in Fig. 6 (a) is a gear train composed of four gears. Since these are circular gears, the pressure ratio of the end driven gear is given by

$$p_3 = \cos\alpha_3 = \frac{d_3}{R_3}$$

Shown in Fig. 6 (b) is the middle gear, where

$$F_2 = \frac{d_3'}{d_2} F_3' = \frac{d_3'}{d_2} F_3$$

Furthermore,

$$p_2 = \frac{F_3}{F_2} p_3 = \frac{d_2}{d_3'} p_3 = \frac{d_2}{R_3'} > \frac{d_2}{R_2} = \cos\alpha_2$$

Therefore, the total pressure ratio p equals $\min\{p_2, p_3\}$. Moreover, the pressure ratio p_2 happened between gears (1) and (2) in this train is greater than their pair pressure ratio $\cos\alpha_2$.

5.2 A Four-Bar Linkage

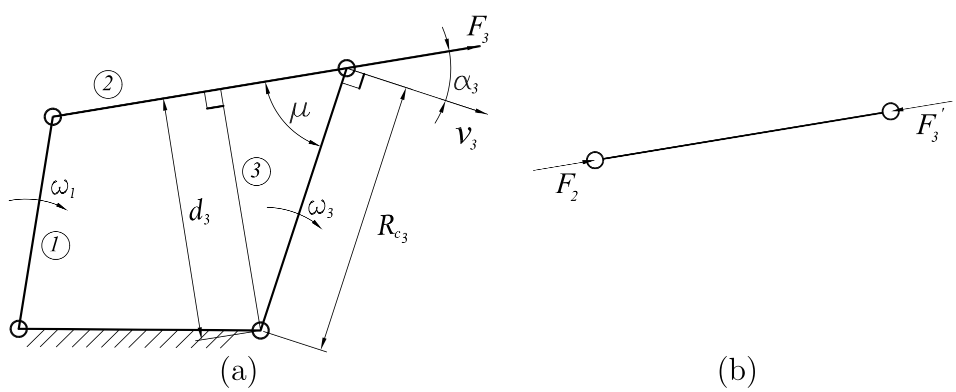


Figure 7: The FBD of (a) the driven link, and (b) the coupler link

As shown in Fig. 7 (a), the pressure ratio of the driven link (3) in the four-bar linkage is given by

$$p_3 = \frac{d_3}{R_{c3}}$$

Since the *action profile* of the driven bar is a circle and the driving force is directed along the joint-centre line of the coupler link, we have a simple relationship:

$$p_3 = \cos\alpha_3 = \sin\mu \tag{4}$$

where α_3 is the deviation angle on the driven link, while μ is the transmission angle of the linkage.

Then, taking the FBD of the link, as shown in Fig. 7 (b), we have $F_2 = F_3' = F_3$. Therefore, $p_2 = p_3$. Consequently, the total pressure ratio p equals both p_2 and p_3. As a result, we obtain only one variable, α_3, μ or p, to assess the transmission quality of this system. This explains why the transmission angle is applied to four-bar linkages. Furthermore, the integration of these three indices is realized by Eq. (4).

5.3 A Cam-Follower Train

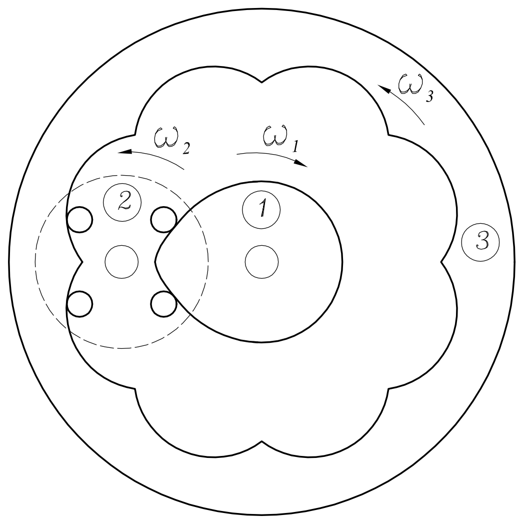
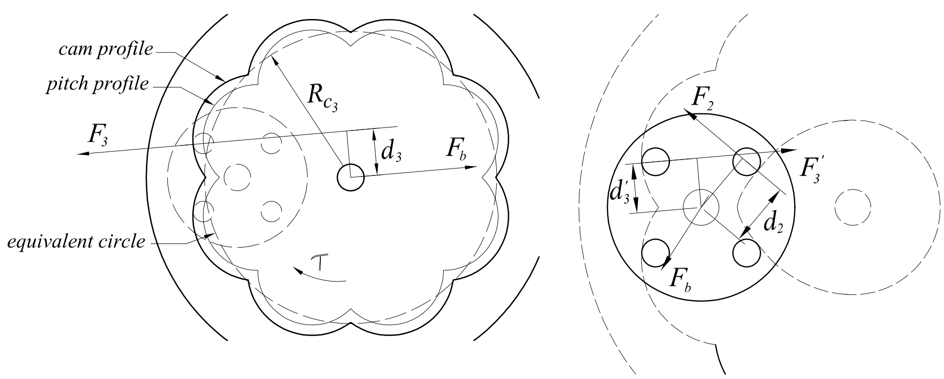


Figure 8: A cam-follower train



(a) (b)

Figure 9: The FBD of (a) the ring cam, and (b) the roller-follower

Shown in Fig. 8 is a cam-follower train composed of a sun cam (1), a ring cam (3) and a roller-carrying disk (2). Consider that the sun cam is the input and the ring cam in the output. From the FBD of the ring cam, as shown in Fig. 9 (a), we have

$$p_3 = \frac{d_3}{R_{c_3}}$$

Then, taking the FBD of the roller-follower as shown in Fig. 9 (b), we obtain,

$$p_2 = \frac{F_3}{F_2} p_3 = \frac{F_3'}{F_2} p_3 = \frac{d_2}{d_3'} p_3$$

Hence, the total pressure ratio p equals $\min\{p_2, p_3\}$, indicating which roller on the disk is the most likely damaged by the contact force.

6 Conclusions

- Since the pressure angle, as an important transmission quality index, fails to evaluate the mechanism systems, whose driven element has non-circular action profile, such as non-circular gear, cam-cam and follower-cam transmission, the pressure ratio, the generalization of the pressure angle, should play an important role in dealing with such systems. Moreover, any mechanism, which can be evaluated by the pressure angle, the transmission angle or the deviation angle, can always be evaluated by the pressure ratio equivalently, so that the terminology complexity may be avoided.

- The driven element determines the pair pressure ratio, while the end driven element is the most important in the analysis of the total pressure ratio in a serial transmission train. The total pressure angle provides a more precise measure than the pressure angle does, because it discovers the worst transmission point.

- The pressure angle and the pressure ratio are applicable only if the direction of the driving force on the driven element is already determined. Furthermore, the former requires the circular action profile of the driven element. The value of the pressure ratio may exceed unity, while the cosine of the pressure angle is bounded within unity.

Acknowledgements

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