IDENTIFYING THE 1-DOF-LOSS VELOCITY-DEGENERATE (SINGULAR) CONFIGURATIONS OF AN 8-JOINT MANIPULATOR

Scott B. Nokleby and Ron P. Podhorodeski
Robotics and Mechanisms Laboratory
Department of Mechanical Engineering, University of Victoria
P. O. Box 3055, Victoria, British Columbia, Canada,V8W 3P6

ABSTRACT

This work presents the determination of the velocity-degenerate (singular) configurations of the National Aeronautics and Space Administration (NASA) Advanced Research Manipulator II (ARMII). A previously developed reciprocity-based methodology for identifying the 1-DOF (degree-of-freedom) loss velocity-degenerate configurations of redundant manipulators is successfully applied to the 8-joint ARMII. It is shown that four sets of conditions (one requiring the satisfaction of a single condition, one requiring the satisfaction of two conditions, and two requiring the satisfaction of three conditions), defining families of degenerate configurations resulting in a single motion DOF loss, exist for manipulators having geometries kinematically equivalent to the ARMII. In addition, reciprocal screws characterizing the lost motion are found for each degenerate configuration. The presented degeneracy conditions are complete and they correct erroneous results previously reported in the literature by other researchers. The results also show that partitioning the matrix of unit joint screw-coordinates to identify velocity-degenerate configurations does not work for redundant spherical-wristed manipulators.

L'IDENTIFICATION DES CONFIGURATIONS DE DÉGÉNÉRESCENCES DES ÉQUATIONS DE VITESSE (SINGULARITÉS) DE PERTE D'UN DEGRÉ-DE-LIBERTÉ D'UN MANIPULATEUR À HUIT JOINTS

RÉSUMÉ

Cet ouvrage présente la détermination des configurations de dégénérescence des équations de vitesse (singularités) du Advanced Research Manipulator II (ARMII) de la National Aeronautics and Space Administration (NASA). Une méthodologie développée antérieurement, basée sur la réciprocité, dans le but d'identifier les configurations de dégénérescence des équations de vitesse d'une perte d'un degré-de-liberté est appliquée avec réussite au ARMII à huit joints. Il est démontré que quatre séries de conditions (une qui exige la réalisation d'une seule condition, une qui exige la réalisation de deux conditions, et deux qui exigent la réalisation de trois conditions), qui définissent les familles de configurations dégénéréscentes qui résultent dans la perte d'un degré-de-liberté, existent pour les manipulateurs avec une géométrie qui est cinématiquement équivalent à celui de l'ARMII. De plus, des visseurs réciproques qui charactèrisent le mouvement perdu sont trouvés pour chaque configuration dégénéréscente. Les conditions dégénéréscentes présentées sont complètes et corrigent des résultats erronés antérieurement annoncés dans la littérature par d'autres chercheurs. Les résultats démontrent également que la partition de la matrice de coordonnées de l'unité visseur pour identifier les configurations de dégénérescence des équations de vitesse ne fonctionne pas pour les manipulateurs redondants à poignet sphérique.

1. INTRODUCTION

The inverse velocity problem of a manipulator, given the desired velocity of the end-effector what are the joint rates (twist amplitudes) required to achieve a desired end-effector velocity, can be solved using screws[1]. For 6-DOF (degree-of-freedom) motion, assuming a non-redundant manipulator in a non-degenerate configuration, the inverse velocity solution can be expressed as:

$$\dot{\boldsymbol{\theta}} = [\$]^{-1}\mathbf{V} \tag{1}$$

where $\dot{\boldsymbol{\theta}}$ is the vector of joint rates, $[\$]$ is the 6x6 matrix of unit joint screw-coordinates (also referred to in the literature as the Jacobian matrix), and $\mathbf{V}$ is the desired end-effector velocity. In a velocity-degenerate configuration, a manipulator loses at least 1-DOF of motion capability, i.e., the joint screws of the manipulator do not span the 6-system of full spatial motion.

The most common method for determining velocity degeneracies of non-redundant manipulators is setting the determinant of the matrix of unit joint screw-coordinates to zero ($\mid [\$] \mid = 0$) to determine the degenerate configurations [1-5].

For redundant manipulators, an infinity of possible solutions exist to the inverse kinematic problem. For a redundant manipulator the matrix of unit joint screw-coordinates is non-square ($[\$]_{6\text{x}n}$ where $n > 6$), therefore, equation (1) cannot be used to solve for the joint rates of a redundant manipulator. Whitney [6] proposed using the Moore-Penrose generalized (pseudo) inverse of $[\$]$ to solve the inverse velocity problem of redundant serial manipulators. The pseudo-inverse of the matrix of unit joint screw-coordinates, $[\$]^+$, is given by:

$$[\$]^+ = [\$]^{\mathrm{T}} \left([\$][\$]^{\mathrm{T}}\right)^{-1} \tag{2}$$

The joint rates can then be found from:

$$\dot{\boldsymbol{\theta}} = [\$]^+\mathbf{V} \tag{3}$$

Numerous other methods have been proposed in the literature to resolve the kinematics of redundant manipulators.

For a redundant manipulator, singularities of the pseudo inverse of $[\$]$ can be examined to resolve velocity-degenerate configurations of redundant manipulators. Velocity-degenerate configurations occur when the determinant of the $[\$][\$]^{\mathrm{T}}$ portion of $[\$]^+$ is equal to zero [7]. Although the matrix formed by $[\$][\$]^{\mathrm{T}}$ is a square matrix, the form of expressions for its elements can be unwieldy. The resulting expression for $\left|[\$][\$]^{\mathrm{T}}\right|$ can be difficult to simplify and analytical solutions to the velocity-degeneracy problem can be hard to find.

Other methods for dealing with the problem of resolving velocity-degenerate configurations of redundantly-actuated serial manipulators have been proposed. Litvin and Parenti Castelli [8] and Litvin et al. [9, 10] used derivatives of displacement functions to form Jacobian matrices of manipulators and considered singularity of the determinants of the Jacobians to identify special configurations. The methodology works for both non-redundant and redundant manipulators.

Podhorodeski, Fenton, and Goldenberg [11] and Podhorodeski, Goldenberg, and Fenton [12, 13] applied a decomposition method to identify the degeneracies of redundant manipulators. The method requires multiple Gram-Schmidt type decompositions to identify all singularities of a redundant manipulator. The proposed method is difficult to apply beyond kinematically-simple (spherical-wristed) redundant manipulators.

[1] See Appendix for a review of manipulator kinematics using screws.

Duffy and Crane III [14], Nokleby and Podhorodeski [15], and Podhorodeski, Nokleby, and Wittchen [16] used 6-joint sub-groups of [$] to determine the velocity-degenerate configurations of redundant manipulators performing a 6-DOF task. Configurations that cause the determinants of all possible 6-joint sub-groups to simultaneously equal zero are velocity-degenerate configurations [17]. This methodology works well for 7-joint manipulators since only seven unique 6-joint sub-groups exist. For an 8-joint manipulator, 28 6-joint sub-groups exist and for a 9-joint manipulator, 84 6-joint sub-groups exist. It is clear that the methodology does not work well for manipulators with higher degrees of redundancy due to the large number of conditions that must be checked to ensure that all the 6-joint sub-group determinants are simultaneously zero.

Kreutz-Delgado, Long, and Seraji [18, 19] used a combination of finding conditions that cause a vector of cofactors of the Jacobian to be zero and looking for row and column dependencies of the Jacobian to determine the velocity-degenerate configurations of 7-joint manipulators.

Burdick [20] developed a recursive algorithm that identifies all singular configurations of revolute-only redundant manipulators. This methodology does not require the formulation of the determinant of [$]. The methodology is based on reciprocity of screws. This is a substantial work, but it has been reported that implementation of the methodology for the symbolic (analytical) case rapidly becomes complex and that identification of velocity-degenerate configurations using numerical results from the algorithm is difficult [21].

Royer, Bidard, and Androit [22] used kinematic geometry to find the velocity-degenerate configurations of a 7-joint anthropomorphic manipulator.

Nokleby and Podhorodeski [23-25] developed a reciprocity-based methodology for finding the 1-DOF-loss velocity-degenerate configurations of kinematically-redundant serial manipulators. A by-product of the methodology is that a reciprocal screw related to the lost motion DOF for each degenerate configuration is determined. Nokleby and Podhorodeski [26, 27] extended their 1-DOF-loss methodology to find multi-DOF-loss velocity-degenerate configurations.

Cheng and Kazerounian [28] determined the singular configurations of the 7-joint anthropomorphic manipulator by studying the manipulator geometrically. They state that a singularity will occur when two revolute joint axes become collinear. This statement is true for non-redundant manipulators, but is not always true for redundant manipulators. The basis of the author's analysis is fundamentally flawed and leads to erroneous statements about the nature of singular configurations of redundant manipulators.

Dupuis [29] and Dupuis, Papadopoulos, and Hayward [30] developed a singular vector method for computing the rank-deficiency loci of rectangular Jacobians. This is a reformulation of the reciprocity-based method of Nokleby and Podhorodeski [23-25] into linear algebra terms. The authors note that the method has an advantage over the reciprocity-based methodology because, in addition to dealing with the case of a Jacobian with more columns than rows (i.e., a redundantly-actuated manipulator), it can handle the case where the Jacobian has more rows than columns. This latter case concerns under-actuated manipulators, i.e., manipulators that have less than the six joints required for 6-DOF spatial motion.

In this paper, the identification of the 1-DOF-loss velocity-degenerate configurations of an 8-joint manipulator, using the reciprocity-based methodology of Nokleby and Podhorodeski [23-25], is considered. The manipulator being analyzed is the National Aeronautics and Space Administration (NASA) Advanced Research Manipulator II (ARMII) [31].

The outline for the remainder of the paper is as follows. In Section 2, the model for the ARMII is presented. In Section 3, the identification of the 1-DOF-loss velocity-degenerate configurations of the ARMII is presented. Section 4 is a discussion of the results. The paper finishes with conclusions.

2. MANIPULATOR MODEL

The ARMII is an 8-joint manipulator with a layout of $(R\perp R\perp R)^{sph}\ \perp R\ \perp(R\perp R\perp R\perp R)^{sph}$. The layout of the ARMII is similar to the 7-joint spherical-revolute-spherical manipulator except the wrist spherical group for the ARMII consists of four joints instead of the three used in the spherical-revolute-spherical manipulator.

The Denavit and Hartenberg (D&H) parameters [32] for the ARMII using Craig's frame assignment convention [33] are presented in Table 1. The parameters of Table 1 correspond to the link transformations:

$$ {}^{j-1}_{j}\mathbf{T} = Rot_{\widehat{\mathbf{x}}_{j-1}}\left(\alpha_{j-1}\right) Trans_{\widehat{\mathbf{x}}_{j-1}}\left(a_{j-1}\right) Trans_{\widehat{\mathbf{z}}_{j}}\left(d_{j}\right) Rot_{\widehat{\mathbf{z}}_{j}}\left(\theta_{j}\right) \tag{4} $$

where ${}^{j-1}_{j}\mathbf{T}$ is a homogeneous transformation describing the location and orientation of link-frame F_j with respect to link-frame F_{j-1}, $Rot_{\widehat{\mathbf{x}}_{j-1}}\left(\alpha_{j-1}\right)$ denotes a rotation about the $\widehat{\mathbf{x}}_{j-1}$ axis by α_{j-1}, $Trans_{\widehat{\mathbf{x}}_{j-1}}\left(a_{j-1}\right)$ denotes a translation along the $\widehat{\mathbf{x}}_{j-1}$ axis by a_{j-1}, $Trans_{\widehat{\mathbf{z}}_{j}}\left(d_{j}\right)$ denotes a translation along the $\widehat{\mathbf{z}}_j$ axis by d_j, and $Rot_{\widehat{\mathbf{z}}_{j}}\left(\theta_{j}\right)$ denotes a rotation about the $\widehat{\mathbf{z}}_j$ axis by θ_j [33]. Figure 1 shows the zero-displacement configuration of the manipulator.

Table 1: Denavit and Hartenberg Parameters for the ARMII Manipulator

F_{j-1}	α_{j-1}	a_{j-1}	d_j	θ_j	F_j
F_0	0	0	0	θ_1	F_1
F_1	$\frac{\pi}{2}$	0	0	θ_2	F_2
F_2	$-\frac{\pi}{2}$	0	g	θ_3	F_3
F_3	$\frac{\pi}{2}$	0	0	θ_4	F_4
F_4	$-\frac{\pi}{2}$	0	h	$\theta_5-\frac{\pi}{2}$	F_5
F_5	$-\frac{\pi}{2}$	0	0	$\theta_6+\frac{\pi}{2}$	F_6
F_6	$\frac{\pi}{2}$	0	0	$\theta_7-\frac{\pi}{2}$	F_7
F_7	$\frac{\pi}{2}$	0	0	θ_8	F_8

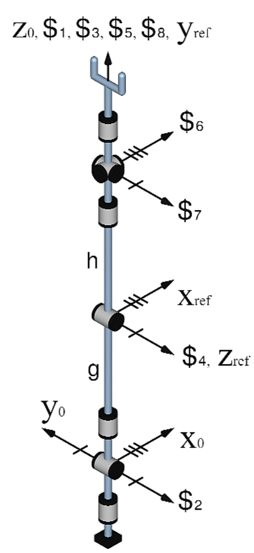

Figure 1: Zero-Displacement Configuration of the ARMII Manipulator

Choosing a reference frame to be an inertial frame coincident with F_4 of the ARMII allows the joint screws to be found as [31]:

$$
\begin{aligned}
{}^{ref}\$_1 &= \left\{ \begin{array}{c} c_2 s_4 + s_2 c_3 c_4 \\ c_2 c_4 - s_2 c_3 s_4 \\ s_2 s_3 \\ -s_2 s_3 \left(c_4 g + h\right) \\ s_2 s_3 s_4 g \\ s_2 c_3 g + h \left(c_2 s_4 + s_2 c_3 c_4\right) \end{array} \right\} \\
{}^{ref}\$_2 &= \left\{ \begin{array}{cccccc} -s_3 c_4, & s_3 s_4, & c_3; & -c_3 \left(c_4 g + h\right), & c_3 s_4 g, & -s_3 \left(g + c_4 h\right) \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_3 &= \left\{ \begin{array}{cccccc} s_4, & c_4, & 0; & 0, & 0, & s_4 h \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_4 &= \left\{ \begin{array}{cccccc} 0, & 0, & 1; & -h, & 0, & 0 \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_5 &= \left\{ \begin{array}{cccccc} 0, & 1, & 0; & 0, & 0, & 0 \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_6 &= \left\{ \begin{array}{cccccc} c_5, & 0, & -s_5; & 0, & 0, & 0 \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_7 &= \left\{ \begin{array}{cccccc} s_5 c_6, & -s_6, & c_5 c_6; & 0, & 0, & 0 \end{array} \right\}^{\mathrm{T}} \\
{}^{ref}\$_8 &= \left\{ \begin{array}{cccccc} -c_5 s_7 + s_5 s_6 c_7, & c_6 c_7, & s_5 s_7 + c_5 s_6 c_7; & 0, & 0, & 0 \end{array} \right\}^{\mathrm{T}}
\end{aligned}
\tag{5}
$$

where c_i and s_i denote $\cos(\theta_i)$ and $\sin(\theta_i)$, respectively. The matrix of unit joint screws for the manipulator is:

$$
{}^{ref}[\$] = {}^{ref}\left[\begin{array}{cccccccc} \$_1 & \$_2 & \$_3 & \$_4 & \$_5 & \$_6 & \$_7 & \$_8 \end{array} \right] \tag{6}
$$

3. IDENTIFICATION OF VELOCITY-DEGENERATE CONFIGURATIONS

Select $\$_2$, $\$_3$, $\$_4$, $\$_5$, $\$_6$, and $\$_7$ from equation (5) to form $[\$]_{sub}$:

$$
{}^{ref}[\$]_{sub} = {}^{ref}\left[\begin{array}{cccccc} \$_2 & \$_3 & \$_4 & \$_5 & \$_6 & \$_7 \end{array} \right] \tag{7}
$$

with the redundant joints being $\$_1$ and $\$_8$. Note that the six joints of $[\$]_{sub}$ were chosen such that they are not inherently linearly dependent. The determinant of $[\$]_{sub}$ is:

$$
\left| {}^{ref}[\$]_{sub} \right| = -c_3 s_4^2 c_6 g h^2 \tag{8}
$$

Therefore, if a) $s_4 = 0$, b) $c_3 = 0$, or c) $c_6 = 0$, then the six joints comprising $[\$]_{sub}$ define a degenerate sub-group of screws. Degenerate configurations of the 8-joint arm will include one of these three conditions. Additional conditions required can be found by enforcing reciprocity of $\$_1$ and $\$_8$ with screws characterizing the lost motion DOF for each of the $[\$]_{sub}$ degenerate conditions.

a) Setting $s_4 = 0$ in equation (7) yields:

$$
{}^{ref}[\$]_{sub_a} = \left[\begin{array}{cccccc} -s_3 c_4 & 0 & 0 & 0 & c_5 & s_5 c_6 \\ 0 & c_4 & 0 & 1 & 0 & -s_6 \\ c_3 & 0 & 1 & 0 & -s_5 & c_5 c_6 \\ -c_3 \left(c_4 g + h\right) & 0 & -h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s_3 \left(g + c_4 h\right) & 0 & 0 & 0 & 0 & 0 \end{array} \right] \tag{9}
$$

The reciprocal screw for the six joints comprising $[\$]_{sub}$ with $s_4 = 0$ can be found from inspection to be:

$$
{}^{ref}\mathbf{W}_{recip_a} = \left\{ \begin{array}{cccccc} 0, & 1, & 0; & 0, & 0, & 0 \end{array} \right\}^{\mathrm{T}} \tag{10}
$$

Note that $\mathbf{W}_{recip_a}$ is not unique. In a 1-DOF-loss degenerate configuration the joint screws span a 5-system and therefore there is an infinity of possible reciprocal screw quantities. These reciprocal screw quantities are all scalar multiples of one another, the one of equation (10) being the case of unit screw-coordinates.

Setting the reciprocal products between $\mathbf{W}_{recip_a}$ & $\$_1$ and $\mathbf{W}_{recip_a}$ & $\$_8$ to zero yields:

$$^{ref}\mathbf{W}_{recip_a} \circledast \ ^{ref}\$_1 = s_2 s_3 s_4 g = 0 \tag{11}$$

$$^{ref}\mathbf{W}_{recip_a} \circledast \ ^{ref}\$_8 = 0 \tag{12}$$

Since $s_4 = 0$, no further conditions are necessary to make $\mathbf{W}_{recip_a}$ reciprocal to joints $\$_1$, $\$_2$, $\$_3$, $\$_4$, $\$_5$, $\$_6$, $\$_7$, and $\$_8$. Therefore, $s_4 = 0$ defines a 1-condition family of degenerate configurations for the 8-joint manipulator.

b) Setting $c_3 = 0$ in equation (7) yields:

$$^{ref}[\$]_{sub_b} = \begin{bmatrix} -s_3 c_4 & s_4 & 0 & 0 & c_5 & s_5 c_6 \\ s_3 s_4 & c_4 & 0 & 1 & 0 & -s_6 \\ 0 & 0 & 1 & 0 & -s_5 & c_5 c_6 \\ 0 & 0 & -h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s_3\left(g + c_4 h\right) & s_4 h & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

The reciprocal screw for the six joints comprising $[\$]_{sub}$ with $c_3 = 0$ can be found from inspection to be:

$$^{ref}\mathbf{W}_{recip_b} = \left\{ \ 0, \quad 1, \quad 0; \quad 0, \quad 0, \quad 0 \ \right\}^{\mathrm{T}} \tag{14}$$

Setting the reciprocal products between $\mathbf{W}_{recip_b}$ & $\$_1$ and $\mathbf{W}_{recip_b}$ & $\$_8$ to zero yields:

$$^{ref}\mathbf{W}_{recip_b} \circledast \ ^{ref}\$_1 = s_2 s_3 s_4 g = 0 \tag{15}$$

$$^{ref}\mathbf{W}_{recip_b} \circledast \ ^{ref}\$_8 = 0 \tag{16}$$

Thus, if $s_2 = 0$ & $c_3 = 0$ or $c_3 = 0$ & $s_4 = 0$, $\mathbf{W}_{recip_b}$ is reciprocal to joints $\$_1$, $\$_2$, $\$_3$, $\$_4$, $\$_5$, $\$_6$, $\$_7$, and $\$_8$. It was shown in Section 3a that $s_4 = 0$ alone results in a degenerate configuration, therefore, $c_3 = 0$ & $s_4 = 0$ does not represent a new family of degenerate configurations. However, $s_2 = 0$ & $c_3 = 0$ defines a new 2-condition family of degenerate configurations.

c) Setting $c_6 = 0$ in equation (7) yields:

$$^{ref}[\$]_{sub_c} = \begin{bmatrix} -s_3 c_4 & s_4 & 0 & 0 & c_5 & 0 \\ s_3 s_4 & c_4 & 0 & 1 & 0 & -s_6 \\ c_3 & 0 & 1 & 0 & -s_5 & 0 \\ -c_3\left(c_4 g + h\right) & 0 & -h & 0 & 0 & 0 \\ c_3 s_4 g & 0 & 0 & 0 & 0 & 0 \\ -s_3\left(g + c_4 h\right) & s_4 h & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Let $\mathbf{W}_{recip_c} = \{ \ L, \quad M, \quad N; \quad P, \quad Q, \quad R \ \}^{\mathrm{T}}$. Setting $^{ref}\mathbf{W}_{recip_c} \circledast \ ^{ref}\$_j = 0$, for $j = 2$ to 7, with $c_6 = 0$ allows the elements of $\mathbf{W}_{recip_c}$ to be found as:

$$^{ref}\mathbf{W}_{recip_c} = \left\{ \ c_5, \quad \frac{c_3 c_4 c_5 - s_3 s_5}{c_3 s_4}, \quad -s_5; \quad s_5 h, \quad 0, \quad c_5 h \ \right\}^{\mathrm{T}} \tag{18}$$

Setting the reciprocal products between $\mathbf{W}_{recip_c}$ & $\$_1$ and $\mathbf{W}_{recip_c}$ & $\$_8$ to zero yields:

$$^{ref}\mathbf{W}_{recip_c} \circledast\ ^{ref}\$_1 = -s_2 s_4 s_5 g = 0 \tag{19}$$

$$^{ref}\mathbf{W}_{recip_c} \circledast\ ^{ref}\$_8 = c_3 s_4 s_6 c_7 h = 0 \tag{20}$$

Note that $s_4 = 0$ or $s_2 = 0$ & $c_3 = 0$ already result in degenerate configurations (see Sections 3a and 3b, respectively) and that $c_3 = 0$, $s_5 = 0$, & $c_6 = 0$ causes $^{ref}\mathbf{W}_{recip_c}$ to collapse into a zero screw ($^{ref}\mathbf{W}_{recip_c} = \mathbf{0}_{6\text{x}1}$), therefore, these conditions do not define new families of degenerate configurations. However, if $s_2 = 0$, $c_6 = 0$, & $c_7 = 0$ or $s_5 = 0$, $c_6 = 0$, & $c_7 = 0$, $\mathbf{W}_{recip_c}$ is reciprocal to joints $\$_1$, $\$_2$, $\$_3$, $\$_4$, $\$_5$, $\$_6$, $\$_7$, and $\$_8$. These sets of conditions define two further 3-condition families of degenerate configurations.

Examining all of the degenerate configurations yields four sets of conditions (one requiring the satisfaction of a single condition, one requiring the satisfaction of two conditions, and two requiring the satisfaction of three conditions) defining families of degenerate configurations resulting in a single motion DOF loss for manipulators having geometries kinematically equivalent to the ARMII. These degenerate configurations and their respective reciprocal screws can be summarized as:

$$\begin{aligned}
&1)\quad s_4 = 0 \\
&\qquad ^{ref}\mathbf{W}_1 =\ ^{ref}\mathbf{W}_{recip_a} = \{\ 0,\ \ 1,\ \ 0;\ \ 0,\ \ 0,\ \ 0\ \}^{\mathrm{T}} \\
&2)\quad s_2 = 0\ \&\ c_3 = 0 \\
&\qquad ^{ref}\mathbf{W}_2 =\ ^{ref}\mathbf{W}_{recip_b} = \{\ 0,\ \ 1,\ \ 0;\ \ 0,\ \ 0,\ \ 0\ \}^{\mathrm{T}} \\
&3)\quad s_2 = 0,\ c_6 = 0,\ \&\ c_7 = 0 \\
&\qquad ^{ref}\mathbf{W}_3 =\ ^{ref}\mathbf{W}_{recip_c} = \{\ c_5,\ \ \tfrac{c_3 c_4 c_5 - s_3 s_5}{c_3 s_4},\ \ -s_5;\ \ s_5 h,\ \ 0,\ \ c_5 h\ \}^{\mathrm{T}} \\
&4)\quad s_5 = 0,\ c_6 = 0,\ \&\ c_7 = 0 \\
&\qquad ^{ref}\mathbf{W}_4 =\ ^{ref}\mathbf{W}_{recip_c} = \{\ c_5,\ \ \tfrac{c_3 c_4 c_5 - s_3 s_5}{c_3 s_4},\ \ -s_5;\ \ s_5 h,\ \ 0,\ \ c_5 h\ \}^{\mathrm{T}}
\end{aligned} \tag{21}$$

The above results were confirmed using the symbolic math package Maple 6 to determine the rank of the matrix of unit joint screw-coordinates, $^{ref}[\$]$. For each set of conditions, the rank of $^{ref}[\$]$ was found to be five, i.e., the manipulator had lost 1-DOF of motion capability.

The reciprocal screw quantity, $\mathbf{W}_i$, in equation (21) characterizes the lost instantaneous motion (velocity) DOF for the i^{th} degenerate configuration. Within the degenerate configuration, the manipulator will not be able to produce a motion that would do work subject to a force spanned by the reciprocal screw. Feasible motions in the i^{th} degenerate configuration are defined by the reciprocal product equation:

$$\mathbf{W}_i \circledast \mathbf{V}_{feasible} = 0 \tag{22}$$

where $\mathbf{V}_{feasible} = \{\boldsymbol{\omega}^{\mathrm{T}};\ \mathbf{v}^{\mathrm{T}}\}^{\mathrm{T}}_{feasible}$ represents the possible instantaneous motions.

4. DISCUSSION

Williams II [31, 34, 35] attempted to derive the velocity-degenerate configurations of the ARMII by looking at a partitioned Jacobian (matrix of unit joint screw-coordinates). Since the ARMII has a spherical wrist, Williams II partitioned the main-arm joints (the joints responsible for translation of the wrist centre) from the spherical-wrist joints (the joints responsible for orientation). The two partitioned groups of joints were analyzed separately to attempt to determine conditions that result in velocity degeneracies for the manipulator.

As Lipkin and Duffy [2] show for a non-redundant manipulator with three revolute-joints forming a spherical group (i.e., intersecting at a common point such as a spherical-wrist centre) that the terms of the screw-coordinate matrix (Jacobian) can be simplified if an appropriate location is used for the reference frame origin. In particular, if the location of the frame of reference is selected to be the intersection point of the three revolute-joint axes, the matrix of unit joint screw-coordinates takes on the form:

$$[\$]_{6\text{x}6} = \begin{bmatrix} [\mathbf{s}_{arm}]_{3\text{x}3} & [\mathbf{s}_{wrist}]_{3\text{x}3} \\ [\mathbf{s}_{o\ arm}]_{3\text{x}3} & \mathbf{0}_{3\text{x}3} \end{bmatrix} \tag{23}$$

where it has been assumed that the final three joints form a spherical wrist. In equation (23), the first three columns are the joint screw-coordinates of the main-arm and the last three columns are the joint screw-coordinates of the spherical-wrist. The determinant of $[\$]_{6\text{x}6}$ is:

$$|[\$]| = \left| \begin{bmatrix} [\mathbf{s}_{arm}] & [\mathbf{s}_{wrist}] \\ [\mathbf{s}_{o\ arm}] & \mathbf{0}_{3\text{x}3} \end{bmatrix} \right| = |[\mathbf{s}_{o\ arm}]|\,|[\mathbf{s}_{wrist}]| \tag{24}$$

Therefore, for a non-redundant serial manipulator, a degeneracy exists if the main-arms joints go degenerate ($|[\mathbf{s}_{o\ arm}]| = 0$) or if the wrist joints go degenerate ($|[\$_{wrist}]| = 0$).

Williams II [31, 34, 35] tries to apply the same principle to redundant manipulators with a spherical-wrist. It should be apparent that this is not possible simply by looking at the matrix of unit joint screw-coordinates for the ARMII. For the chosen frame of reference, the ARMII matrix of unit joint screw-coordinates is of the form:

$$[\$]_{6\text{x}8} = \begin{bmatrix} [\mathbf{s}_{arm}]_{3\text{x}4} & [\mathbf{s}_{wrist}]_{3\text{x}4} \\ [\mathbf{s}_{o\ arm}]_{3\text{x}4} & \mathbf{0}_{3\text{x}4} \end{bmatrix} \tag{25}$$

where the first four columns are the joint screw-coordinates of the main-arm and the last four columns are the joint screw-coordinates of the spherical-wrist. The matrix of unit joint screw-coordinates for the ARMII is not square, therefore, no determinant can be taken. Noting that the pseudo inverse of the matrix of unit joint screw-coordinates is $[\$]^+ = [\$]^\mathrm{T}\left([\$][\$]^\mathrm{T}\right)^{-1}$, the determinant of $[\$][\$]^\mathrm{T}$ can be used to determine velocity-degenerate configurations [7]. The matrix $[\$][\$]^\mathrm{T}$ is of the form:

$$\begin{aligned} [\$][\$]^\mathrm{T} &= \begin{bmatrix} [\mathbf{s}_{arm}] & [\mathbf{s}_{wrist}] \\ [\mathbf{s}_{o\ arm}] & \mathbf{0}_{3\text{x}4} \end{bmatrix} \begin{bmatrix} [\mathbf{s}_{arm}]^\mathrm{T} & [\mathbf{s}_{o\ arm}]^\mathrm{T} \\ [\mathbf{s}_{wrist}]^\mathrm{T} & \mathbf{0}_{4\text{x}3} \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{s}_{arm}][\mathbf{s}_{arm}]^\mathrm{T} + [\mathbf{s}_{wrist}][\mathbf{s}_{wrist}]^\mathrm{T} & [\mathbf{s}_{arm}][\mathbf{s}_{o\ arm}]^\mathrm{T} \\ [\mathbf{s}_{o\ arm}][\mathbf{s}_{arm}]^\mathrm{T} & [\mathbf{s}_{o\ arm}][\mathbf{s}_{o\ arm}]^\mathrm{T} \end{bmatrix} \end{aligned} \tag{26}$$

Comparing the matrices of equations (23) and (26), the $[\$][\$]^\mathrm{T}$ matrix does not have the sub-matrix of zeros that is found in the matrix of unit joint screw-coordinates for the non-redundant spherical-wristed manipulator. It is this sub-matrix of zeros that allows the partitioning of the main-arm and wrist joints to allow the determination of velocity-degenerate configurations of non-redundant spherical-wristed manipulators. It is clear from the form of $[\$][\$]^\mathrm{T}$ that looking at conditions that cause only the main-arm or the wrist to go singular for redundant spherical-wristed manipulators does not guarantee that the whole manipulator is in a velocity-degenerate configuration.

The degeneracy conditions Williams II [34, 35] found can be summarized as: a) $s_4 = 0$, b) $s_2 = 0$ & $c_3 = 0$, c) $s_2 = 0$ & $s_4 = 0$, and d) $c_6 = 0$ & $c_7 = 0$. Williams II [34] points out that condition (c) is a subset of (a), i.e., condition (c) is a special case of the more general condition (a). Therefore, Williams II identified three unique sets of conditions resulting in velocity degeneracies for the ARMII: i) $s_4 = 0$, ii) $s_2 = 0$ & $c_3 = 0$, iii) $c_6 = 0$ & $c_7 = 0$. Conditions (i) and (ii) agree with the results obtained using the reciprocity-based methodology. Condition (iii) does not agree with the results obtained using the reciprocity-based methodology. Williams II [34, 35] claims that condition (iii) is a degeneracy for the full Jacobian, but this is not correct. It can be shown that with $c_6 = 0$ & $c_7 = 0$ the Jacobian retains full rank (this was confirmed using Maple 6). The results of the reciprocity-based methodology show that the condition $c_6 = 0$ & $c_7 = 0$ alone does not result in a degeneracy, but requires either $s_2 = 0$ or $s_5 = 0$ to be true in addition. The results show that partitioning the matrix of unit joint screw-coordinates to identify singular configurations does not work for redundant spherical-wristed manipulators.

5. CONCLUSIONS

A reciprocity-based methodology for identifying the 1-DOF-loss velocity-degenerate (singular) configurations of redundant manipulators was successfully applied to the 8-joint NASA ARMII manipulator. It was shown that four sets of conditions (one requiring the satisfaction of a single condition, one requiring the satisfaction of two conditions, and two requiring the satisfaction of three conditions) define families of degenerate configurations resulting in a single motion DOF loss for manipulators having geometries kinematically equivalent to the ARMII. In addition, reciprocal screws characterizing the lost motion were found for each degenerate configuration. The presented degeneracy conditions are complete and they correct erroneous results previously reported in the literature by other researchers.

It was also shown that partitioning the matrix of unit joint screw-coordinates to identify velocity-degenerate configurations does not work for redundant spherical-wristed manipulators.

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APPENDIX A: MANIPULATOR KINEMATICS USING SCREWS

Scews

A screw ($\mathbf{S}$) is a line in space having an associated linear pitch. Screws can be represented as:

$$\mathbf{S} = \left\{ \begin{array}{c} \mathbf{s} \\ \mathbf{s}_o \end{array} \right\} = \lambda \left\{ \begin{array}{c} \mathbf{l} \\ \mathbf{l}_o + p\mathbf{l} \end{array} \right\} \tag{27}$$

where $\mathbf{s}$ and $\mathbf{s}_o$ are the screw-coordinates, $\mathbf{l}$ and $\mathbf{l}_o$ are the Plücker coordinates of the line, λ is an associated magnitude, and p is the pitch of the screw [5, 36]. The pitch of a screw can be found from:

$$p = \frac{\mathbf{s} \cdot \mathbf{s}_o}{\|\mathbf{s}\|}$$

A screw is said to be a unit or normalized screw ($) if $\|\mathbf{s}\| = 1$ or in the case where $\mathbf{s} = \mathbf{0}_{3\text{x}1}$, if $\|\mathbf{s}_o\| = 1$ [5].

For manipulators, a revolute joint can be represented by a zero-pitch unit screw $\$_{rev} = \{\mathbf{l}^{\mathrm{T}};\, \mathbf{l}_o^{\mathrm{T}}\}^{\mathrm{T}}$ and a prismatic joint can be represented by an infinite-pitch unit screw which when normalized to the ∞-pitch gives $\$_{pris} = \{\mathbf{0}_{3\text{x}1}^{\mathrm{T}};\, \mathbf{l}^{\mathrm{T}}\}^{\mathrm{T}}$.

The screw-coordinates of the joints of a manipulator can be found from:

$$^{ref}\$_j = \left\{ \begin{array}{c} ^{ref}\widehat{\mathbf{z}}_j \\ ^{ref}\widehat{\mathbf{z}}_j \times {}^{ref}\mathbf{r}_{j \to p_{ee}} \end{array} \right\}_{revolute} \quad \text{or} \quad = \left\{ \begin{array}{c} \mathbf{0}_{3\text{x}1} \\ ^{ref}\widehat{\mathbf{z}}_j \end{array} \right\}_{prismatic} \tag{28}$$

where $\widehat{\mathbf{z}}_j$ denotes the unit vector of the j^{th} joint axis direction, p_{ee} denotes a point coincident with the origin of F_{ref} and attached to the end-effector, and $\mathbf{r}_{j \to p_{ee}}$ denotes a vector from a point on the axis of joint j to the point p_{ee}.

Scew Transformations

A screw transformation is defined as:

$$^{ref_a}_{ref_b}\mathbf{T}_{\mathbf{S}} = \left[\begin{array}{cc} ^{ref_a}_{ref_b}\mathbf{R} & \mathbf{0}_{3\text{x}3} \\ ^{ref_a}\widetilde{\mathbf{p}}_{o_a \to o_b}\, {}^{ref_a}_{ref_b}\mathbf{R} & ^{ref_a}_{ref_b}\mathbf{R} \end{array} \right] \tag{29}$$

where $^{ref_a}_{ref_b}\mathbf{R}$ is a 3x3 rotation matrix and $^{ref_a}\widetilde{\mathbf{p}}_{o_a \to o_b}$ is a 3x3 cross product skew-symmetric matrix based on the components of a vector, $^{ref_a}\mathbf{p}_{o_a \to o_b} = {}^{ref_a}\{p_{x_{o_a \to o_b}}, p_{y_{o_a \to o_b}}, p_{z_{o_a \to o_b}}\}^{\mathrm{T}}$, from the origin of F_{ref_a} to the origin of F_{ref_b}. The cross product skew-symmetric matrix $^{ref_a}\widetilde{\mathbf{p}}_{o_a \to o_b}$ is defined as:

$$^{ref_a}\widetilde{\mathbf{p}}_{o_a \to o_b} = \left[\begin{array}{ccc} 0 & -p_{z_{o_a \to o_b}} & p_{y_{o_a \to o_b}} \\ p_{z_{o_a \to o_b}} & 0 & -p_{x_{o_a \to o_b}} \\ -p_{y_{o_a \to o_b}} & p_{x_{o_a \to o_b}} & 0 \end{array} \right] \tag{30}$$

Reciprocal Scews

Let the screw quantity $\mathbf{A} = \{\mathbf{a}^{\mathrm{T}};\, \mathbf{a}_o^{\mathrm{T}}\}^{\mathrm{T}}$ represent the velocity of a body and the screw quantity $\mathbf{B} = \{\mathbf{b}^{\mathrm{T}};\, \mathbf{b}_o^{\mathrm{T}}\}^{\mathrm{T}}$ represent a wrench acting on the body. If wrench $\mathbf{B}$ contributes nothing to the rate of work being done to the body moving with velocity $\mathbf{A}$, the screw quantities $\mathbf{A}$ and $\mathbf{B}$ are said to be reciprocal to one another [36]. Mathematically, the two screws, **A** and **B**, are reciprocal if their reciprocal product is zero:

$$\mathbf{A} \circledast \mathbf{B} = \mathbf{a} \cdot \mathbf{b}_o + \mathbf{a}_o \cdot \mathbf{b} = 0 \tag{31}$$

where $\circledast$ denotes a reciprocal product between two screws.

Velocity Solutions

The end-effector velocity ($\mathbf{V}$) of a manipulator is defined as:

$$\mathbf{V} = \left\{ \begin{array}{c} \boldsymbol{\omega} \\ \mathbf{v} \end{array} \right\} \tag{32}$$

where $\boldsymbol{\omega}$ is the angular velocity and $\mathbf{v}$ is the translational velocity. The forward velocity solution of a serial manipulator, given the joint rates (twist amplitudes) what is the end-effector velocity, can be described using screws. Letting $\$_j$ denote the screw-coordinates of joint j allows $\mathbf{V}$ to be determined by:

$$\mathbf{V} = \left\{ \begin{array}{c} \boldsymbol{\omega} \\ \mathbf{v} \end{array} \right\} = \sum_{j=1}^{n} \$_j \dot{\theta}_j \tag{33}$$

where n is the total number of joints and $\dot{\theta}_j$ is the joint rate (twist amplitude) of joint j. In matrix form this can be expressed as:

$$\mathbf{V} = \left\{ \begin{array}{c} \boldsymbol{\omega} \\ \mathbf{v} \end{array} \right\} = [\$]\dot{\boldsymbol{\theta}} \tag{34}$$

where $[\$] = \left[\begin{array}{cccc} \$_1 & \$_2 & \cdots & \$_n \end{array} \right]$ is the matrix of unit joint screw-coordinates commonly referred to as the Jacobian matrix and $\dot{\boldsymbol{\theta}} = \left[\begin{array}{cccc} \dot{\theta}_1 & \dot{\theta}_2 & \cdots & \dot{\theta}_n \end{array} \right]^{\mathrm{T}}$ is the vector of joint rates.

For 6-DOF (degree-of-freedom) motion, assuming a non–velocity-degenerate and non-redundant manipulator (i.e., $n = 6$), the inverse velocity solution can be expressed as:

$$\dot{\boldsymbol{\theta}} = [\$]^{-1}\mathbf{V} \tag{35}$$

Force Solutions

The wrench ($\mathbf{F}$) applied by the end-effector of a manipulator is defined as:

$$\mathbf{F} = \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\} \tag{36}$$

where $\mathbf{f}$ is the force applied and $\mathbf{m}$ is the moment applied. The inverse static force problem of a serial manipulator, given the wrench being applied by the end-effector what are the joint torques (or forces for prismatic joints), can be solved using conservation of power (power in equals power out):

$$\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = \{\mathbf{f}^{\mathrm{T}}; \mathbf{m}^{\mathrm{T}}\} \left\{ \begin{array}{c} \mathbf{v} \\ \boldsymbol{\omega} \end{array} \right\} = \mathbf{F}^{\mathrm{T}}\left([\Delta]\,\mathbf{V}\right) \tag{37}$$

where $\boldsymbol{\tau}$ is a vector of joint torques and forces and the matrix $[\Delta]$ is an interchange operator that transforms screws between axis-coordinate order to ray-coordinate order and is defined as:

$$[\Delta] = \left[\begin{array}{cc} \mathbf{0}_{3\mathrm{x}3} & \mathbf{I}_{3\mathrm{x}3} \\ \mathbf{I}_{3\mathrm{x}3} & \mathbf{0}_{3\mathrm{x}3} \end{array} \right] \tag{38}$$

Substituting the forward velocity solution $\mathbf{V} = [\$]\dot{\boldsymbol{\theta}}$ into equation (37) yields:

$$\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = \mathbf{F}^{\mathrm{T}}\,[\Delta]\,[\$]\dot{\boldsymbol{\theta}} \tag{39}$$

Equation (39) is true for all $\dot{\boldsymbol{\theta}}$, therefore:

$$\boldsymbol{\tau}^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\,[\Delta]\,[\$] \tag{40}$$

Transposing both sides of equation (40) yields the inverse static force problem:

$$\boldsymbol{\tau} = [\$]^{\mathrm{T}}\,[\Delta]\,\mathbf{F} \tag{41}$$

where the fact that $[\Delta]^{\mathrm{T}} = [\Delta]$ has been utilized.

The forward static force solution for a non-redundant manipulator can be expressed as:

$$\mathbf{F} = \left([\$]^{\mathrm{T}}\,[\Delta]\right)^{-1}\boldsymbol{\tau} = [\Delta]\,[\$]^{-\mathrm{T}}\boldsymbol{\tau} \tag{42}$$

where the fact that $[\Delta]^{-1} = [\Delta]$ has been utilized.