Integrated Type and Approximate Dimensional Synthesis of Four-Bar Planar Mechanisms for Rigid Body Guidance

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In this paper a combination of geometric and numerical methods is used to combine type and approximate dimensional synthesis of planar four-bar mechanisms for rigid body guidance. The developed algorithm sizes link lengths, locates joint axes, and decides between RR- and PR-dyads that, when combined, guides a rigid body through the best approximation of n specified poses (positions and orientations), where $n \geq 5$. No initial guesses of type or dimension are required.

The synthesis of a planar four-bar mechanism that can guide a rigid-body exactly through five finitely separated poses is known as the five-position Burmester problem [1]. Five poses define a finite number of four-bar mechanisms. When $n \geq 5$ the system of synthesis equations is overconstrained, and in general no exact solution exists. The problem then is to find a four-bar mechanism that can guide a rigid-body through the n poses with the least amount of error. Furthermore, dimensional synthesis for rigid body guidance generally assumes a mechanism type: i.e., planar $4R$; slider-crank; crank-slider; trammel, etc. This method generalizes approximate mechanism synthesis by integrating type and dimensional synthesis.

An equation of a line or circle can be expressed:

$$\mathbf{CK} = \begin{bmatrix} x^2+y^2 & 2x & 2y & 1 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \mathbf{0}, \quad (1)$$

where x and y are the Cartesian coordinates of points on a circle or line, and the K_i define the geometry [2]. For a circle,

$$\begin{aligned} K_0 &= 1, \\ K_1 &= -X_c, \\ K_2 &= -Y_c, \\ K_3 &= K_1^2 + K_2^2 - r^2, \end{aligned} \quad (2)$$

defines a circle of radius r centred at (X_c, Y_c). For a line,

$$\begin{aligned} K_0 &= 0, \\ \frac{K_1}{K_2} &= -\tan\vartheta, \\ K_3 &= x\sin\vartheta - y\cos\vartheta, \end{aligned} \quad (3)$$

defines a linear range of points (x, y) that make an angle ϑ with the positive x-axis. Three points are necessary to define a circle, and two points for a line. For n points, where n is greater than three for a circle and two for a line, the system becomes overconstrained, and a least squares approximation is necessary to determine the best fit. To set up the problem, the row vector on the left hand side of Equation (1) becomes an $n \times 4$ matrix. The problem is solved using singular value decomposition.

A variant of singular value decomposition (SVD) factors any given $m \times n$ matrix $\mathbf{C}$ into

$$\mathbf{C}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^T_{n\times n}, \quad (4)$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal and $\mathbf{S}$ is an diagonal matrix containing the singular values of $\mathbf{C}$ in descending order. For a least squares approximation of overconstrained systems, the last column of $\mathbf{V}$ is the best approximation of $\mathbf{K}$ such that $\mathbf{CK} = \mathbf{0}$. The $\mathbf{K}$ parameter vector defining the geometry is then any scalar multiple of that column of $\mathbf{V}$. For convenience, the solution vector is normalized by the first parameter, in order to match the solution with the parameters defining a circle, as given in Equation (2).

If the geometry of the identified circle appears to be inordinately large, the geometry may instead be determined using Equation (3) to define a line, as a line segment is analogous to a circular arc of infinite radius centred at infinity. Fitting data points to circles and lines is the basis of integrated type and approximate dimensional synthesis using this method.

Suppose that n planar poses of a rigid body are to be approximated, such that $n > 5$. Suppose also that the linkage shown in Figure 1 best approximates the rigid body motion defined by the n poses. For the linkage shown, the motion of reference frame E defines the

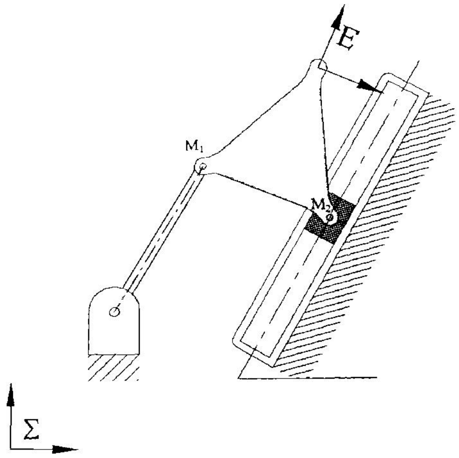

Figure 1: A linkage that best approximates $n > 5$ poses.

rigid body motion with respect to the grounded coordinate frame Σ. Frame E is related to frame Σ by a translation of (a, b) and a rotation of θ. The points of the rigid body in frame Σ can be found by transforming the same points in frame E, which are known to be constant. The transformation is

$$\begin{bmatrix} x_\Sigma \\ y_\Sigma \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_E \\ y_E \\ 1 \end{bmatrix}, \tag{5}$$

where (x_Σ, y_Σ) is a point in frame Σ, (x_E, y_E) is the same point frame E, and (a, b) and θ define the transformation from frame Σ to frame E. For this method, we are interested in determining the locations of joints M_1 and M_2. In frame E, the coordinate system that moves with the coupler, the positions of the joints are constant. However, in frame Σ, joint M_1 is bound to a circle, and joint M_2 to a line. If we have n positions of M_1 and M_2 in Σ, the geometry of the circle and line may be found by singular value decomposition.

In order to determine the positions of the joints in Σ, it is first necessary to find the positions of the joints in E, as the two are related by Equation (5). A property of **C** in Equation (1) is that it approximates either a line or circle. The more linearly dependent its rows are with one another, the closer it approximates a line or circle. Therefore, one can choose values of (x, y) to make the rows of **C** the most linearly dependent, thus making **C** the most ill-conditioned. The problem then becomes a 2-dimensional search.

The conditioning of a matrix can be measured by the ratio of its largest and smallest singular values, which is called the condition number κ.

$$\kappa \equiv \frac{\sigma_{MAX}}{\sigma_{MIN}}, 1 \leq \kappa \leq \infty \tag{6}$$

A more convenient number to use is the inverse of the condition number γ, with $0 \leq \gamma \leq 1$, because it is bounded in both directions. An ill-conditioned matrix has $\gamma \approx 0$. Also, the closer **C** is to being singular, the closer the **K** parameters are to defining an exact circle or line. Therefore, the goal is to find x and y such that γ is minimized.

The Nelder-Mead polytope algorithm may be used for this minimization [3]. Since this algorithm needs as input an initial guess of the parameters it is searching for, γ or κ may be plotted in terms of x and y first, and approximate values are chosen that minimize γ. At least two minima are required to obtain a planar four-bar mechanism, as each minimum corresponds to a single dyad. The Nelder-Mead algorithm is then fed these parameters as inputs, and determines the values of x and y that give the smallest values of γ.

Once the values of x and y have been determined, the set of values of x_Σ and y_Σ can then be solved for. The **K** parameters may then be found using singular value decomposition. The distinction between RR and PR dyads is found by determining whether the resulting K parameters better describe a circle or line. A resulting circle defines an RR dyad, while a line defines a PR dyad. If using Equation (2) on the **K** parameters defines a circle having geometry several orders of magnitude greater than the poses, it is recalculated using Equation (3) to define a line instead. In this case, it is defined as a PR, rather than an RR.

This method has been verified by several means. It has been tested using rigid-body motion of known planar four-bar mechanisms of all types, with and without induced noise. It has also been tested with rigid-body motion that cannot be reproduced by planar four-bar mechanisms. Both types of testing reveal the robustness of the method to noise, and show its ability to synthesize mechanisms that approximate motion no planar four-bar mechanism could replicate exactly.

References

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