Wrench-Closure Workspace of Six-DOF Parallel Mechanisms Driven by 7 Cables

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Abstract

The wrench-closure workspace (WCW) of six-degree-of-freedom (DOF) parallel cable-driven mechanisms is defined as the set of poses of the moving platform of the mechanism for which any external wrench can be balanced by tension forces in the cables. This workspace is fundamental in order to analyze and design parallel cable-driven mechanisms. This paper deals with the class of six-DOF mechanisms driven by seven cables. Two theorems, which provide efficient means to test whether a given pose of the moving platform belongs to the WCW, are proposed. One of these two theorems reveals the nature of the boundary of the constant-orientation cross sections of the WCW. Moreover, some of the possible applications of these theorems are discussed and illustrated.

Espace des Configurations Polyvalentes des Mécanismes Parallèles à Six Degrés de Liberté Actionnés à l'aide de Sept Câbles

Résumé

L'espace des configurations polyvalentes des mécanismes parallèles à six degrés de liberté actionnés à l'aide de câbles est défini comme l'ensemble des poses de la plate-forme du mécanisme pour lesquelles n'importe quel torseur externe peut être équilibré par des forces de tension dans les câbles. Cet espace est fondamental pour l'analyse et la conception des mécanismes parallèles actionnés à l'aide de câbles. Cet article traite des mécanismes à six degrés de liberté actionnés par sept câbles. Deux théorèmes, permettant de tester efficacement si une pose de la plate-forme appartient à l'espace des configurations polyvalentes, sont proposés. Un de ces deux théorèmes révèle la nature des frontières des coupes a orientation constante de l'espace des configurations polyvalentes. Des applications possibles de ces théorèmes sont également discutées et illustrées.

1 Introduction

The mobile platform of the cable-driven mechanisms studied in this paper is connected in parallel to a base by seven lightweight links such as cables or wires. The base contains actuated reels for the storage, extension and retraction of the cables. Each cable has its own reel. For instance, a parallel mechanism driven by seven cables is shown in Figure 2. By controlling the length of their respective cables, the actuated reels allow the control of the six degrees of freedom (DOF) of the mobile platform. Compared to rigid-link parallel mechanisms, the use of lightweight links allows a reduction of the overall mass and inertia of the mechanism. However, since a cable can only pull and not push on the mobile platform, the forces applied by the cables on the platform have a unidirectional nature. Therefore, the relationship between the pose (position and orientation) and the feasible wrenches at the platform is an important issue for parallel cable-driven mechanisms. For instance, in [1] and [2] the workspace of planar parallel cable-driven mechanisms is studied as the set of poses of the moving platform for which a particular wrench can be generated at the platform by pulling on it with the cables. In [3] and [4], the nature and the determination of another workspace, called the wrench feasible workspace, is discussed. This workspace is defined as the set of poses of the moving platform for which any wrench of a given set of wrenches can be balanced with tension forces in the cables. In the present paper, the workspace is defined as the set of poses of the mobile platform for which any wrench can be generated at the platform by tightening the cables. This workspace is called the wrench-closure workspace (WCW) and has been studied by the authors in [5] in the case of three-DOF planar parallel mechanisms driven by four cables. This work is an extension of the study undertaken in [5] to six-DOF parallel mechanisms driven by seven cables.

A class of six-DOF parallel mechanisms driven by six cables have been investigated in [6] and [7]. The WCW of such six-cable-driven mechanisms does not exist since, for any given non-zero wrench that can be generated with taut cables, the generation of the opposite of this wrench requires that the six cables push on the platform and, consequently, this opposite wrench is not feasible. Indeed, a necessary condition on the number of cables for the WCW of six-DOF mechanisms to exist is: the number of cables must be greater than the number of DOF of the platform, i.e., greater than six. References [8] and [9] introduce the design of an eight-cable mechanism, the WARP mechanism, which is intended for high speed assembling in [8] and to create virtual sensation of motion in [9]. Application of six-DOF parallel mechanisms driven by seven cables to high speed manipulation is also discussed in [10]. Reference [11] proposes a geometrical approach to design six-DOF cable-driven mechanisms intended to serve in wind tunnels as active suspension devices. Other examples of application of cable-driven mechanisms are a force display system [12] and a multi-finger haptic interface device [13]. Additionally, an index to evaluate the force transmission characteristics of n-DOF parallel mechanisms driven by $n+1$ cables was introduced in [14].

From a general design point of view, since no specific task is assigned to the cable-driven mechanism, the WCW is of great interest. As the shape and the size of the WCW are highly dependent on the architecture—coordinates of the attachment points of the cables on the platform and locations of the actuated reels on the base—of the cable-driven mechanism, its determination is an important tool to analyze and devise such mechanisms. But, to the best of our knowledge, although it has its own importance, no specific study of the WCW of six-DOF parallel mechanisms has been yet undertaken. The present paper aims mainly at providing necessary and sufficient conditions that furnish an efficient means to determine if a pose of the moving platform belongs to the WCW. To this end, two theorems—Theorem 3 and Theorem 4—are proposed. Moreover, one of these theorems reveals the nature of the boundary of the constant-orientation cross sections of the WCW. Examples

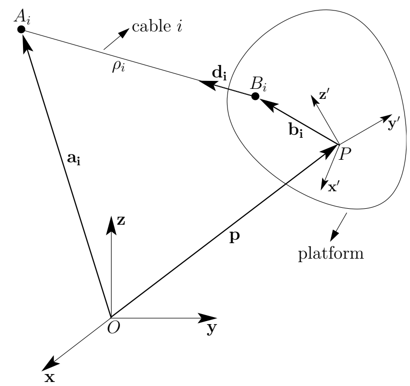

Figure 1: Kinematic modeling.

of applications of these theorems are also presented.

2 Kinematic Modeling and Wrench Transmission

2.1 Modeling and Notations

As shown in Figure 1, let us consider a fixed reference frame $(O, \mathbf{x}, \mathbf{y}, \mathbf{z})$ attached to the base of the cable-driven mechanism, referred to as the base frame, and a moving frame $(P, \mathbf{x}', \mathbf{y}', \mathbf{z}')$ attached to the mobile platform, where P is the reference point to be positioned by the mechanism. The orientation of the moving frame with respect to the base frame describes the orientation of the mobile platform with respect to the base of the mechanism. The point A_i, at which the ith cable $(i = 1, 2, \ldots, 7)$ winds around its reel, is assumed to be fixed relative to the base. Furthermore, the ith cable is attached at point B_i on the mobile platform and this attachment point is assumed to be fixed relative to the mobile platform. The ith cable is tense between the points A_i and B_i and assumed to be a segment of the straight line $(A_i B_i)$, its taut length is denoted ρ_i. The contact points A_i and B_i are modeled as spherical joints. Then, let us denote $\mathbf{a}_i$ and $\mathbf{b}_i$ the vectors $\overrightarrow{OA_i}$ and $\overrightarrow{PB_i}$, respectively, in the base frame. The position $\mathbf{p} = [x, y, z]^T$ of the mobile platform is given by vector $\overrightarrow{OP}$ in the base frame. When expressed in the base frame, the unit vector along cable i, $(1/\rho_i)\overrightarrow{B_iA_i}$, is denoted $\mathbf{d}_i$. Its expression is

$$\mathbf{d}_i = (\mathbf{a}_i - \mathbf{b}_i - \mathbf{p})/\rho_i = (\mathbf{a}_i^v - \mathbf{p})/\rho_i \tag{1}$$

with

$$\mathbf{a}_i^v = \mathbf{a}_i - \mathbf{b}_i \tag{2}$$

where $\mathbf{a}_i^v$ is the position vector of the point A_i^v. This vector depends on the mechanism architecture and on its orientation. When $\mathbf{p} = \mathbf{a}_i^v$, $\rho_i = 0$—B_i and A_i coincide—and Eq. (1) is no longer valid. In this case, it is convenient to define $\mathbf{d}_i$ as the zero vector.

2.2 Wrench Matrix

The taut cable i exerts at B_i a pure force $t_i\mathbf{d}_i$ on the mobile platform, where t_i is the tension in the cable. By definition, t_i is always nonnegative. This pure force generates a moment $\mathbf{b}_i \times t_i\mathbf{d}_i$ at the reference point P of the mobile platform and the wrench (force/moment pair) applied at P by the ith cable is $t_i\mathbf{w}_i$, with wrench $\mathbf{w}_i$ defined by

$$\mathbf{w}_i = \begin{bmatrix} \mathbf{d}_i \\ \mathbf{b}_i \times \mathbf{d}_i \end{bmatrix}. \tag{3}$$

If $\mathbf{w}_p$ denotes the wrench applied at P by the seven cables of the mechanism, since $\mathbf{w}_p$ is the sum of the cable wrenches $t_i\mathbf{w}_i$, the relationship between the tensions t_i in the cables and the wrench $\mathbf{w}_p$ can be written in matrix form as

$$\mathbf{W}\mathbf{t} = \mathbf{w}_p \tag{4}$$

with

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_7 \end{bmatrix} \tag{5}$$

and

$$\mathbf{t} = \begin{bmatrix} t_1 & t_2 & \dots & t_7 \end{bmatrix}^T \tag{6}$$

where $\mathbf{t}$ is the vector of cable tensions and $\mathbf{W}$ the 6×7 pose dependent wrench matrix. Henceforth, for any vector $\mathbf{v}$, $\mathbf{v} > \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$ and $\mathbf{v} < \mathbf{0}$ mean that all the components of $\mathbf{v}$ are greater than zero, greater than or equal to zero and smaller than zero, respectively.

3 Two Characterizations of the WCW

3.1 Definition and Null Space Characterization of the WCW

According to the modeling presented in the previous section, let us state a precise definition of the WCW.

Definition 1 *The WCW is the set of poses of the mobile platform where, for any wrench $\mathbf{w}_p$ in $\mathbb{R}^6$, there exists at least one vector $\mathbf{t} \geq \mathbf{0}$, $\mathbf{t} \in \mathbb{R}^7$, such that $\mathbf{W}\mathbf{t} = \mathbf{w}_p$.*

It is noted that, from a general point of view, the WCW is a six-dimensional subset of $\mathbb{R}^6$. Now, let us state a fundamental theorem that provides a null space characterization of the WCW.

Theorem 1 *A pose belongs to the WCW if and only if*

$$\text{rank}(\mathbf{W}) = 6 \tag{7}$$

and

$$\exists\, \mathbf{z} \in \ker(\mathbf{W}) \ \textit{such that}\ \mathbf{z} > \mathbf{0} \tag{8}$$

where $\ker(\mathbf{W})$ stands for the null space of $\mathbf{W}$. A proof of this theorem can be found in [15] and [16]. Note that, in [15], the WCW is known as the set of fully constrained configurations. In [16], a mechanism is said to be manipulable when its pose belongs to the WCW, whereas, in [10], when the wrench matrix $\mathbf{W}$ satisfies the two conditions of Theorem 1, the set formed by its columns is called a “Vector Closure”. Note also that Eq. (8) can be given the following interpretation: a zero wrench can be applied at reference point P of the moving platform—$\mathbf{w}_p = \mathbf{0}$— by tightening the seven cables of the mechanism.

Then, in order to establish a clear link between Theorem 1 and a set of poses of the moving platform, the following theorem is fundamental.

Theorem 2 *If* $\mathbf{W}$ *has full rank then* $\ker(\mathbf{W}) = \operatorname{span}(\mathbf{z}_0)$ *with*

$$\mathbf{z}_0 = \begin{bmatrix} \det([\ \mathbf{w}_7\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_6\ ]) \\ \det([\ \mathbf{w}_1\ \ \mathbf{w}_7\ \ \dots\ \ \mathbf{w}_6\ ]) \\ \vdots\ \ \ddots \\ \det([\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_7\ ]) \\ -\det([\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_6\ ]) \end{bmatrix}_7 . \tag{9}$$

Let us prove Theorem 2. Since $\mathbf{W}$ has full rank, its null space $\ker(\mathbf{W})$ is a one-dimensional subspace of $\mathbb{R}^7$. Consequently, for any non-zero vector $\mathbf{z}$ of the null space of $\mathbf{W}$

$$\ker(\mathbf{W}) = \operatorname{span}(\mathbf{z}). \tag{10}$$

Now, since $\mathbf{W}$ has full rank, the vector $\mathbf{z}_0$ defined by Eq. (9) is not null and we may assume that $\det([\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_6\ ]) \neq 0$, i.e., that $(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_6)$ is a basis of $\mathbb{R}^6$. Hence

$$\exists\ (\beta_1, \beta_2, \dots, \beta_6) \in \mathbb{R}^6 \mid \mathbf{w}_7 = \sum_{j=1}^{6} \beta_j \mathbf{w}_j \tag{11}$$

but with Eq. (9)

$$\begin{aligned} \mathbf{W}\mathbf{z}_0 \ &= \ \sum_{i=1}^{6} \mathbf{w}_i \det([\ \mathbf{w}_1\ \ \dots\ \ \mathbf{w}_{i-1}\ \ \mathbf{w}_7\ \ \mathbf{w}_{i+1}\ \ \dots\ \ \mathbf{w}_6\ ]) \\ &\quad -\mathbf{w}_7 \det([\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_6\ ]) \end{aligned} \tag{12}$$

The use of Eq. (11) and of classic properties of determinants leads to

$$\begin{aligned} & \det([\ \mathbf{w}_1\ \ \dots\ \ \mathbf{w}_{i-1}\ \ \mathbf{w}_7\ \ \mathbf{w}_{i+1}\ \ \dots\ \ \mathbf{w}_6\ ]) \\ = \ & \det([\ \mathbf{w}_1\ \ \dots\ \ \mathbf{w}_{i-1}\ \ \sum_{j=1}^{6} \beta_j \mathbf{w}_j\ \ \mathbf{w}_{i+1}\ \ \dots\ \ \mathbf{w}_6\ ]) \\ = \ \beta_i\ & \det([\ \mathbf{w}_1\ \ \dots\ \ \mathbf{w}_{i-1}\ \ \mathbf{w}_i\ \ \mathbf{w}_{i+1}\ \ \dots\ \ \mathbf{w}_6\ ]) \end{aligned}$$

$\forall\ i$, $1 \leq i \leq 6$. Hence with Eq. (12)

$$\mathbf{W}\mathbf{z}_0 = \det([\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \dots\ \ \mathbf{w}_6\ ]) \left(\sum_{i=1}^{6} \beta_i \mathbf{w}_i - \mathbf{w}_7 \right). \tag{13}$$

Finally, from Eq. (11)

$$\mathbf{W}\mathbf{z}_0 = \mathbf{0} \tag{14}$$

i.e. $\mathbf{z}_0$ belongs to the nullspace of $\mathbf{W}$ and, since $\mathbf{z}_0$ is not null, $\ker(\mathbf{W}) = \text{span}(\mathbf{z}_0)$ and this concludes the proof of Theorem 2. Now, since vector $\mathbf{z}_0$ is a non-zero vector if and only if $\mathbf{W}$ has full rank, a consequence of theorems 1 and 2 is:

Theorem 3 *A pose belongs to the WCW if and only if* $\mathbf{z}_0 > \mathbf{0}$ *or* $\mathbf{z}_0 < \mathbf{0}$*.*

Theorem 1 can be directly used to test if a pose of the moving platform of a parallel cable-driven mechanism belongs to the WCW. However, in order to check whether Eq. (8) is true or not, an optimization method is used to find if there exists a vector $\mathbf{z}$ in the null space of $\mathbf{W}$ which satisfies the constraint $\mathbf{z} > \mathbf{0}$. To test if a pose is inside the WCW, Theorem 3 is much more efficient than Theorem 1 since it does not require the use of an optimization method, but requires only the test of the signs of seven determinants. Moreover, Theorem 3 brings insight into the nature and into some properties of the WCW.

3.2 On the Nature of the Boundary of the WCW

Since the determinants in Eq. (9) are continuous functions of the pose of the mobile platform, if $\det_i$ denotes the determinant of the square matrix obtained from the wrench matrix $\mathbf{W}$ by deleting its ith column and z_{0i} denotes the ith component of vector z_0, $\det_i = (-1)^i z_{0i} = 0$ is the equation of a hypersurface embedded in the six-dimensional space of the poses of the mobile platform. Furthermore, if they exist, the parts of this hypersurface on which all the other determinants appearing in Eq. (9) have the same sign, i.e., on which all the components but component i of vector $\mathbf{z}_0$ have the same sign, are potential parts of the boundary of the WCW. Finally, according to Theorem 2 and to Theorem 3, the boundary of the WCW is composed of parts of such hypersurfaces.

When the mobile platform is kept at a constant orientation, the set of all the poses of the moving platform which belong to the WCW is called the constant-orientation WCW (COWCW). Hence, by definition, the COWCW is a subset of the three-dimensional Cartesian space. Now, if the ith cable of the mechanism is removed, and, if the other six cables are replaced by a kinematic chain composed of a universal joint, an actuated prismatic joint and a spherical joint, a six-DOF parallel manipulator commonly known as the Gough-Stewart (GS) platform is obtained. The so-called Jacobian matrix of this GS platform turns out to be the transpose of the square matrix obtained from the wrench matrix $\mathbf{W}$ by deleting its ith column. Thus, $z_{0i} = (-1)^i \det_i = 0$ is the equation of the (type II) singularity locus of this GS platform, and, according to [17], z_{0i} can be written as a multivariate polynomial of degree three in the Cartesian coordinates x, y and z of the moving platform. This polynomial, referred to as a cubic surface, represents the constant-orientation singularity locus of the GS platform. Now, according to Theorem 3, a part of this cubic surface associated with z_{0i} on which all the other components z_{0j} $(j \neq i)$ of vector $\mathbf{z}_0$ have the same sign is a possible part of the boundary of the COWCW. Moreover, according again to Theorem 3, *the boundary of the COWCW consists of parts of cubic surfaces of the same nature as the constant-orientation (type II) singularity locus of the GS platform.*

3.3 Another Characterization of the WCW

Let us state a theorem that provides another characterization of the WCW.

Theorem 4 *For any integer* i, $1 \leq i \leq 7$, *chosen arbitrarily, a pose belongs to the WCW if and only if*

$$\det(\mathbf{W}_i) \neq 0 \tag{15}$$

and

$$-\mathbf{W}_i^{-1}\mathbf{w}_i > \mathbf{0} \tag{16}$$

with the definition

$$\mathbf{W}_i = \begin{bmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_{i-1} & \mathbf{w}_{i+1} & \dots & \mathbf{w}_7 \end{bmatrix}_{6\times 6}. \tag{17}$$

Note that Eq. (16) means that all the six components of vector $-\mathbf{W}_i^{-1}\mathbf{w}_i$ are greater than zero. Now, in order to prove Theorem 4, let us assume that a given pose of the mobile platform belongs to the WCW. According to Theorem 3, none of the seven components of vector $\mathbf{z}_0$ defined by Eq. (9) is equal to zero. Since the ith component of $\mathbf{z}_0$, z_{0i}, is equal to $(-1)^i \det(\mathbf{W}_i)$, then, for all integer i $(1 \leq i \leq 7)$, $\det(\mathbf{W}_i) \neq 0$ and Eq. (15) of Theorem 4 is true. Then, let us prove that Eq. (16) is also verified. From Eq. (8), for all i $(i = 1, 2, \dots, 7)$

$$\mathbf{W}_i\mathbf{z}_i = -\mathbf{w}_i z_i \tag{18}$$

where $\mathbf{z}_i$ is the six-dimensional column vector obtained from $\mathbf{z}$ by deleting its ith component z_i. Since Eq. (15) has already been proved to be true, $\det(\mathbf{W}_i) \neq 0$ for all i and Eq. (18) is equivalent to

$$\mathbf{z}_i = -\mathbf{W}_i^{-1}\mathbf{w}_i z_i. \tag{19}$$

According to Eq. (8), z_i and all the components of vector $\mathbf{z}_i$ are greater than zero. Therefore, for all i

$$-\mathbf{W}_i^{-1}\mathbf{w}_i > \mathbf{0} \tag{20}$$

meaning that Eq. (16) is true. Thus, if a pose of the mobile platform belongs to the WCW, then, Eq. (15) and Eq. (16) of Theorem 4 are verified. Then, let us demonstrate the reverse implication. Let us assume that the two conditions of Theorem 4 are true. Obviously, Eq. (15) implies Eq. (7). Then, let $\mathbf{z}^*$ be the vector $-\mathbf{W}_7^{-1}\mathbf{w}_7$. According to Eq. (16), $\mathbf{z}^* > \mathbf{0}$, and, by definition of $\mathbf{z}^*$

$$\mathbf{W}_7\mathbf{z}^* = -\mathbf{w}_7 \tag{21}$$

thus

$$\begin{bmatrix} \mathbf{W}_7 & \mathbf{w}_7 \end{bmatrix} \begin{bmatrix} \mathbf{z}^* \\ 1 \end{bmatrix} = \mathbf{0} \tag{22}$$

i.e.

$$\mathbf{W}\mathbf{z} = \mathbf{0} \tag{23}$$

with

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}^* \\ 1 \end{bmatrix} > \mathbf{0}. \tag{24}$$

Hence, Eq. (8) of Theorem 1 is true. In conclusion, Eq. (15) and Eq. (16) of Theorem 4 are equivalent to Eq. (7) and Eq. (8) of Theorem 1, and, consequently, Theorem 4 is true. Note that the theorems

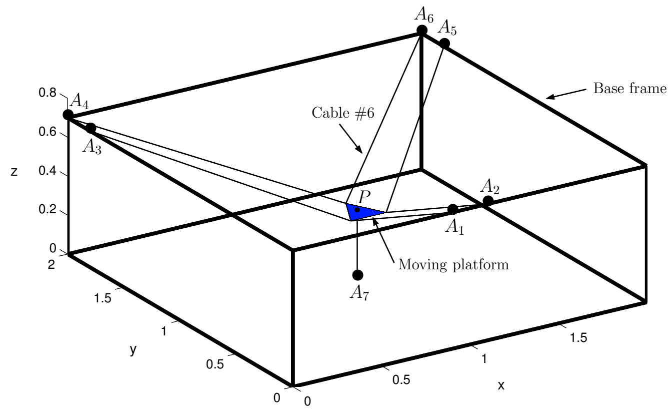

Figure 2: A 6-DOF parallel mechanism driven by seven cables for its reference orientation.

1 and 4 are stated for six-DOF parallel mechanisms driven by seven cables. However, since the proofs of these theorems do not depend on the dimension of the vector spaces involved, by a suitable modification of the size and number of matrices and vectors appearing in the demonstration of this subsection, Theorem 1 and 4 can be proved to be equivalent for any n-DOF parallel mechanism driven by $n+1$ cables. Finally, since Eq. (16) requires a matrix inversion, in order to deal with a well-conditioned problem, the subscript i involved in Eq. (16) must be chosen such that

$$\kappa(\mathbf{W}_i) \leq \kappa(\mathbf{W}_j),\ \forall\, j \neq i,\ 1 \leq j \leq 7 \tag{25}$$

where $\kappa(\mathbf{W}_i)$ denotes the condition number of $\mathbf{W}_i$ for all integers i.

4 Examples of Application of the Determination of the WCW

In order to illustrate some possible applications of theorems 3 and 4, let us consider the six-DOF seven-cable-driven parallel mechanism shown in Figure 2. Its moving platform is a triangle, i.e., the attachment points B_i $(i = 1, 2, \ldots, 6)$ are coincident by pairs. The seventh cable of the mechanism is attached at the reference point P which coincides with the centroid of the triangle.

4.1 Determination of the COWCW

The basic steps of an algorithm that produces a wire-frame type representation of the boundary of the COWCW are now described.
Step 1: choose an orientation for the moving platform.
Step 2: create a cloud of points by means of a discretization of a box which contains the points A_i $(i = 1, 2, \ldots, 7)$ of the base of the cable-driven mechanism.
For instance, the box chosen for the mechanism shown in Figure 2 corresponds to its base frame.
Step 3: use Theorem 3 or Theorem 4 to test each point of the cloud of points. Keep the points which correspond to poses of the moving platform belonging to the COWCW and form a set $\mathcal{S}_1$ with them. If $\mathcal{S}_1$ is the empty set, according to this algorithm, the COWCW does not exist and the algorithm is stopped.

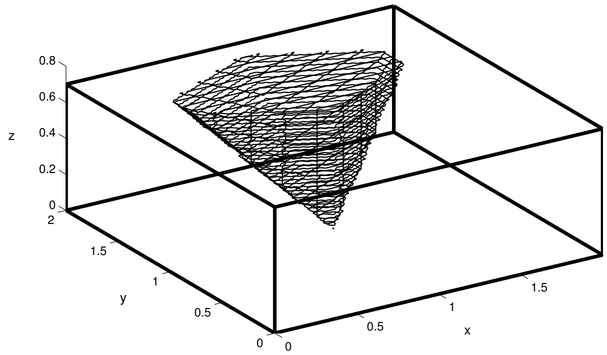

Figure 3: The COWCW of the mechanism shown in Figure 2 in its reference orientation.

Note that, if the resolution of the cloud of points created at Step 2 is too coarse, the COWCW may be found to be non-existent although it may actually exist.
Step 4: among the points of $\mathcal{S}_1$, find those which belong to the boundary of $\mathcal{S}_1$ and form a set $\mathcal{S}_2$ with them.
The points that belong to the boundary of $\mathcal{S}_1$ are the points of $\mathcal{S}_1$ which do not have all their six nearest neighbours belonging to $\mathcal{S}_1$.
Step 5: create two sets, $\mathcal{P}_1$ and $\mathcal{P}_2$, of parallel planes such that each plane of $\mathcal{P}_1$ is orthogonal to each plane of $\mathcal{P}_2$, and, such that some of the planes of $\mathcal{P}_1$ and $\mathcal{P}_2$ contain the points belonging to the set $\mathcal{S}_2$.
For instance, the wire-frame type representation of a COWCW shown in Figure 3 has been obtained with the planes of $\mathcal{P}_1$ parallel to the xy-plane and with the planes of $\mathcal{P}_2$ parallel to the xz-plane.
Step 6: in each of the planes of $\mathcal{P}_1$ and $\mathcal{P}_2$ which contain points of $\mathcal{S}_2$, link the neighbouring points of $\mathcal{S}_2$ with straight lines. Produce a wire-frame type representation of the COWCW by drawing these straight lines.
Note that the resolution of the cloud of points must be sufficiently fine for Step 6 to yield good results. This algorithm has been applied to obtain the COWCW of the seven-cable mechanism shown in Figure 2. The orientation chosen in Step 1 is the orientation of the moving platform shown in Figure 2. The result is shown in Figure 3. If a zyz-convention of Euler angles is used to describe the orientation of the moving platform of the mechanism shown in Figure 2, where ϕ, θ and ψ are the angles of the convention, Figure 4 shows the COWCW computed by the algorithm for $\phi = -\pi/12$, $\theta = -\pi/12$ and $\psi = 0$.

4.2 Numerical Approximation of the Volume

Theorem 3 and Theorem 4 can also be applied to obtain numerical approximations of the volume of COWCWs. A method, proposed in [18] and based on a discretization of the Cartesian space, has been programmed. For instance, an approximation of the volume of the COWCW shown in Figure 3 is 0.17629 m^3—the base frame having a volume equal to 2.8 m^3. Figure 5 shows that the volume of the COWCW of the mechanism shown in Figure 2 decreases abruptly when the orientation of its moving platform changes. In Figure 5, the aforementioned zyz-convention of Euler angles has been chosen to describe the orientation of the moving platform and the angles ϕ and θ are associated with

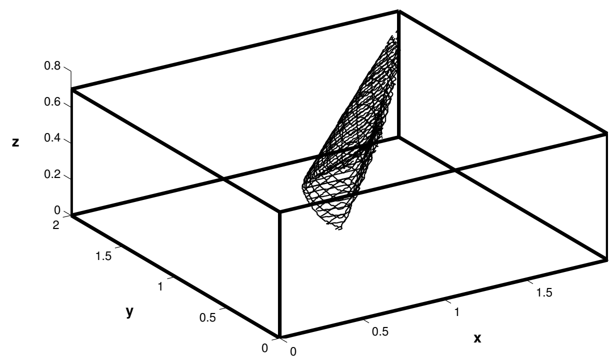

Figure 4: The COWCW of the mechanism shown in Figure 2 for $\phi = -\pi/12$, $\theta = -\pi/12$ and $\psi = 0$.

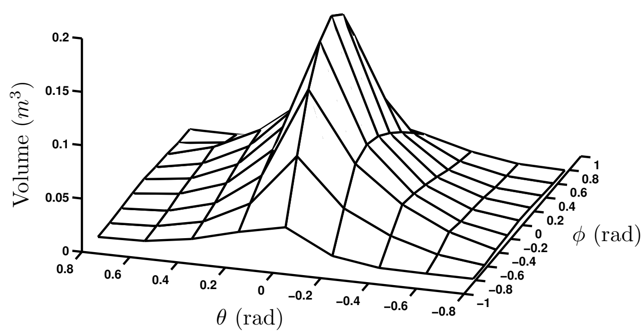

Figure 5: Volume of the COWCW for different orientations of the mobile platform of the mechanism shown in Figure 2, with $\psi = 0$.

the first two rotations of the convention, respectively. The third angle ψ is assumed to be equal to zero.

4.3 Dexterous WCW

Defining the dexterous WCW as the set of positions of the moving platform that belong to the WCW for any orientation of the moving platform within a given set of orientations $\mathcal{O}$, Theorem 3 or Theorem 4 can be used to determine the dexterous WCW. Briefly, a range of orientations is discretized to form the set $\mathcal{O}$ and an initial cloud of points is created. For the first orientation of $\mathcal{O}$, the points of the initial cloud of points are tested and those which do not belong to the COWCW are deleted. The remaining points are then tested for the second orientation of $\mathcal{O}$ and deleted if they do not belong to the COWCW. This procedure is repeated until no points remain or until all the orientations of $\mathcal{O}$ have been considered. In the latter case, the dexterous WCW exists and a graphical representation can be obtained. For example, taking a zyz-convention of Euler angles for the mechanism shown in Figure 2 and choosing a range of orientations such that the first and second angle of the zyz-convention belong to $[-\pi/12; \pi/12]$ and the third angle is equal to zero, the dexterous WCW obtained is shown in Figure 6. This figure illustrates how the WCW of seven-cable

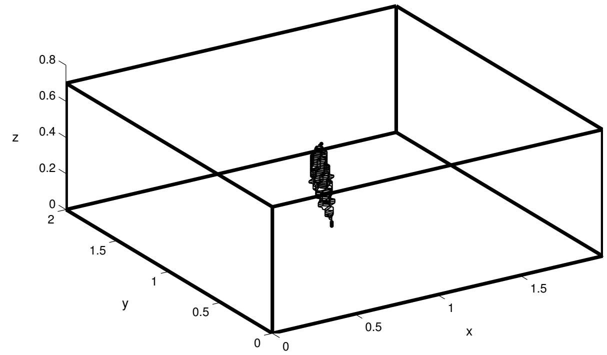

Figure 6: A dexterous workspace of the mechanism shown in Figure 2.

mechanisms is a small workspace. Note that the dexterous WCW shown in Figure 6 can be enlarged by increasing the dimensions of the moving platform of the mechanism shown in Figure 2. However, the size of the new dexterous WCW obtained remains small in comparision with the dimensions of the base frame.

5 Conclusion

Two theorems, both allowing to test efficiently if a pose of the moving platform of a six-DOF seven-cable-driven parallel mechanism belongs to the WCW, have been proposed. One of these two theorems reveals that the boundary of the constant-orientation cross sections of the WCW consists of parts of cubic surfaces of the same nature as the constant-orientation singularity locus of the Gough-Stewart platform. These theorems are fundamental tools for the analysis and design of six-DOF parallel mechanisms driven by seven cables. Moreover, the results presented in this paper form the necessary basis, on one hand, to the study of the WCW of six-DOF parallel mechanisms driven by more than seven cables and, on the other hand, to the development of algorithms which determine the WCW by taking advantage of the geometric nature of its boundary and which are, consequently, much more efficient than discretization algorithms based on clouds of points.

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