Discrete-time Stability and Vibrations of Systems With Unidirectional Force Control

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Abstract – A single degree-of-freedom (DoF) mechanical model is considered to analyze the stability of unidirectional digital force control that makes it possible to establish an analytical solution procedure. The stability of the proportional-derivative (PD) force control strategy is discussed. The frequency of the arising vibrations along the stability limits are also determined as a function of the sampling frequency of the digital controller. The stability charts clearly show the range of applicability of the digital PD force control in practical tasks.

1 Introduction

It is a well-known phenomenon that the digitally controlled robots have vibration problems when they are in contact with the environment and force control is applied on them. The control design of these robotic systems usually employs continuous-time approaches and models. Recently published books about robotic force control [1, 2] present investigations only for the case of continuous-time force control of robotic manipulators. In addition, basic control textbooks (like [3, 4]) usually determine the sampling period of a digital controller with the help of continuous time arguments. Using the frequency response bandwidth or the crossover frequency of the continuous-time system, these books suggest to determine an appropriate (sufficiently small) sampling period for the discrete time realization. Certainly, these rules of thumb work properly with most of the systems. However, in case of a rigid mechanical system (e.g. an industrial robot touching a turbine blade) with a small effective damping in the force controlled direction, these rules do not ensure stability. Experiments show that the digital realization of analog control algorithms often leads to instability, and the digitally controlled system starts to oscillate at a relatively low frequency [5]. It depends on the control and mechanical parameters of the system whether a control algorithm with a certain sampling frequency can be considered continuous, or the digital effects have an essential role in the dynamic behavior.

In this paper, the discrete-time stability and vibrations of a single DoF robot model with digital PD force control is presented. Particular attention has been paid to the effects of the derivative gain to investigate whether derivative feedback can or should be included in the force control of robots. Results are given in the form of stability charts in the parameter space of the sampling time, the control gains and the mechanical parameters.

2 Mechanical model

Figure 1 shows a single DoF model that can serve to study the behavior of a robotic arm with unidirectional force control. The equivalent mass m and stiffness k represent the inertia and stiffness of the robot and the environment in the force controlled direction. The generalized force Q represents the effects of the joint drives, while C denotes the magnitude of the effective Coulomb friction force. These parameters can be calculated using the constraint Jacobian representing the force controlled direction, and the mass and stiffness matrices of the robot [6].

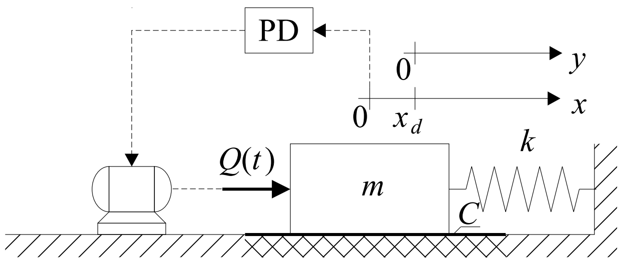

Figure 1: Single DoF model of a force controlled robot

Thus, considering a PD force controller, the equation of motion of the model shown in Figure 1 has the form

$$\begin{aligned} m\ddot{x}(t) + C\,\mathrm{sgn}\dot{x}(t) + kx(t) &= Q(t) \\ Q(t) &= F_m(t) - P(F_m(t) - F_d) - D\dot{F}_m(t) \end{aligned} \tag{1}$$

where P and D are the proportional and derivative feedback gains. In addition, $F_m(t) = kx(t)$ denotes the measured force, while $F_d = kx_d$ stands for the desired contact force. Variable y denotes a small perturbation around the desired position x_d is to be used later in Section 3.

3 Stability analysis

For the case of continuous-time force control, it can easily be shown that the model presented in Figure 1 always results in an asymptotically stable behavior for any positive gains $P > 0$, $D > 0$. The steady state force error can be calculated by the simple formula $\Delta F = C/P$. Thus, the higher the proportional gain is the smaller the force error becomes. On the other hand, a PD digital force control for the same model can have a very different behavior, where the stability and dynamic performance characteristics are determined by the control gains, the sampling frequency and the mechanical parameters of the system.

According to [3, 4] and [5] the stability analysis of the discrete-time system can be carried out via the construction of a discrete map of the state variables. For this calculation, we neglect dry friction from the model, and a zero-order-hold (ZOH) is used to model the sampling of the digital controller. In addition, a perturbation y is introduced around the desired equilibrium position x_d with $x(t) = x_d + y(t)$. Then, using the state variable vector $\mathbf{y} = [y,\ \dot{y}]^T$, the equation of motion (1) for the nth sampling period can be rewritten in the form

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{y}((n-1)h)\ , \quad t \in [nh, (n+1)h)\ ,\ n = 0, 1, 2, \ldots \tag{2}$$

where h is the sampling time of the digital controller and the coefficient matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_\mathrm{n}^2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ (1-P)\omega_\mathrm{n}^2 & -D\omega_\mathrm{n}^2 \end{bmatrix} . \tag{3}$$

Here, the natural angular frequency of the uncontrolled system is denoted by $\omega_\mathrm{n} = \sqrt{k/m}$. Introduce also the notation $t_n = nh$ for the nth sampling instant and let the 4 dimensional "discrete" state vector be $\mathbf{z}_{n+1} = [\mathbf{y}_{n+1}, \mathbf{y}_n]^T$. Then, the solution of equation (2) yields the discrete mapping between consecutive states as follows

$$\begin{gathered} \mathbf{z}_{n+1} = \mathbf{W}\mathbf{z}_n\ , \quad \mathbf{W} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Gamma} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \text{where} \\ \boldsymbol{\Phi} = e^{\mathbf{A}h} = \begin{bmatrix} \cos(\omega_\mathrm{n}h) & \frac{1}{\omega_\mathrm{n}}\sin(\omega_\mathrm{n}h) \\ \omega_\mathrm{n}\sin(\omega_\mathrm{n}h) & \cos(\omega_\mathrm{n}h) \end{bmatrix} , \\ \boldsymbol{\Gamma} = \int_0^h e^{\mathbf{A}\tau} d\tau \mathbf{B} = \begin{bmatrix} (1-\cos(\omega_\mathrm{n}h))(1-P) & -(1-\cos(\omega_\mathrm{n}h))D \\ \omega_\mathrm{n}\sin(\omega_\mathrm{n}h)(1-P) & -\omega_\mathrm{n}\sin(\omega_\mathrm{n}h)D \end{bmatrix} , \end{gathered} \tag{4}$$

and $\mathbf{I}$ is the identity matrix. The substitution of the standard exponential trial solution $\mathbf{z}_n = \mathbf{c}e^{\lambda n}$ into the difference equation (4) yields the characteristic polynomial

$$(\mathbf{I}e^{\lambda} - \mathbf{W})\mathbf{c} = \mathbf{0} \quad \Rightarrow \quad p_4(\mu) = \det(\mathbf{I}\mu - \mathbf{W}) \tag{5}$$

where $\mathbf{c}$ is a kind of discrete vibration mode vector and $\mu = \exp(\lambda)$ is the so-called characteristic multiplier. By expanding $p_4(\mu)$, and considering the expressions of equation (4), the characteristic equation can be obtained as

$$a_0\mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu = 0\ , \tag{6}$$

with the coefficients

$$\begin{aligned} a_0 &= 1\ , \\ a_1 &= -2\cos(\omega_\mathrm{n}h)\ , \\ a_2 &= D\omega_\mathrm{n}\sin(\omega_\mathrm{n}h) - (1-P)(1-\cos(\omega_\mathrm{n}h)) + 1\ , \\ a_3 &= -D\omega_\mathrm{n}\sin(\omega_\mathrm{n}h) - (1-P)(1-\cos(\omega_\mathrm{n}h))\ . \end{aligned} \tag{7}$$

Obviously, the zero trivial solution of equation (2), which corresponds to the desired contact force of the control task, is asymptotically stable if and only if all the roots of the characteristic equation (6) are inside the unit circle of the complex plane. This can directly be checked by Jury's test, which is the discrete-time analog of the well known Routh-Hurwitz criterion [3]. The resulting stability criteria are

$$\begin{aligned} c_1 =& p_4(1) = a_0 + a_1 + a_2 + a_3 = 2(1-\cos(\omega_\mathrm{n}h))P > 0\,, \\ c_2 =& (-1)^4 p_4(-1) = a_0 - a_1 + a_2 - a_3 = 2 + 2\cos(\omega_\mathrm{n}h) + 2\sin(\omega_\mathrm{n}h)\omega_\mathrm{n} D > 0\,, \\ c_3 =& 1 - a_2 + a_3 a_1 - a_3^2 = -(1-\cos(\omega_\mathrm{n}h))^2 P^2 + (2d\sin(\omega_\mathrm{n}h) - 4\cos(\omega_\mathrm{n}h) + 1)\times \\ & \times (1-\cos(\omega_\mathrm{n}h))P + 2(\cos(\omega_\mathrm{n}h) - d\sin(\omega_\mathrm{n}h))(d\sin(\omega_\mathrm{n}h) + 3(1-\cos(\omega_\mathrm{n}h))) > 0\,. \end{aligned} \tag{8}$$

There are four physical parameters in (8) that can be collected into three independent dimensionless variables. One is the proportional gain P, second is expression $\omega_\mathrm{n}h/(2\pi)$, which is just the ratio of the natural frequency of the uncontrolled system and the sampling frequency 1/h, and third is the dimensionless derivative gain $d = D\omega_\mathrm{n}$. Using these parameters, the stability charts corresponding to the conditions $c_i > 0$, $i = 1, 2, 3$ are shown in Figure 2.

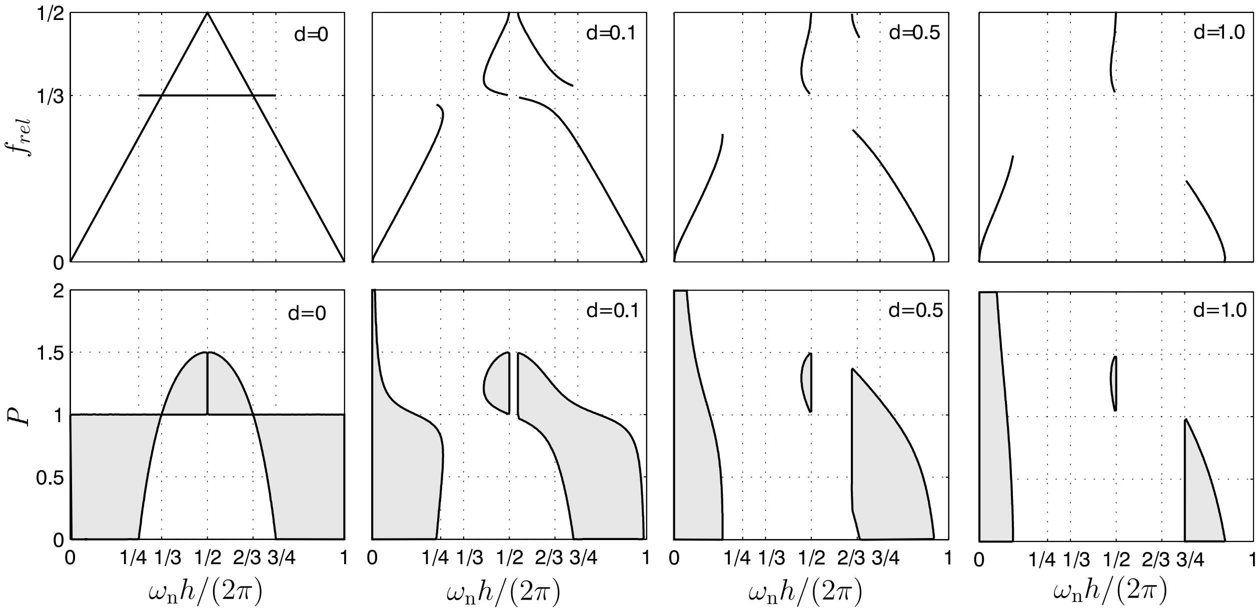

Figure 2: Stability and vibration frequency charts (stable domains are shaded)

According to this figure and recalling that the continuous-time system with PD force controller is stable for any positive gain values, the main difference between the continuous- and discrete-time controllers is apparent. The stable domain of the control parameters is no longer infinite and the maximal proportional gain within the limits of stability depends on the frequency ratio $\omega_\mathrm{n}h/(2\pi)$ and the applied differential gain d. The disjoint structure of these charts illustrates well the intricate dynamic behavior of discrete time systems and shows that there are some ranges of mechanical parameters where no stable control is possible.

In addition, the frequencies of the vibrations developed along the stability boundaries are calculated numerically and presented with respect to the sampling frequency above the corresponding stability chart. The ratio of the frequency of possible arising vibrations and the sampling frequency is called the relative vibration frequency and it is denoted by f_{rel}. Figure 2 shows that $f_{rel} \leq 1/2$ for all the values of the control gains. Thus, in case of losing stability, the system will start to oscillate with a relatively low vibration frequency. The frequency of possible arising vibrations will be always less than the half of the sampling frequency.

4 Conclusions

In this paper, the discrete-time stability and the arising vibrations of mechanical systems subjected to digital force control were investigated with closed form calculations. The discrete-time nature of the force control causes an intricate dynamic behavior. Many undesired events in force controlled systems (e.g. instability, low frequency oscillations) are caused by the discrete-time nature of the controller. It has been known that the proportional gains can have finite values only in the presence any finite sampling time. The literature provided upper estimations for these sampling periods without paying attention to the possible periodic nature of the stability charts with respect to the frequency ratio of the sampling and the natural frequencies.

By adding a derivative gain, the proportional gain can be further increased within the limits of stability to minimize the steady state force error. However, because the stable domain is very narrow for high values of the proportional gain, the system becomes very sensitive for the relative tuning of the sampling and natural frequencies and can easily lose stability if small variations occur in the parameters. The stability chart (see Figure 2) actually brakes up to disjoint domains. These are the important effects of the derivative gain.

The derivative gain is commonly used in position control, but it has usually been avoided in force control without any analytical explanation. This paper gives analytical results which show that the use of a derivative gain reduces the area of the stable domains in certain regions of the stability charts. This can be the main reason why the derivative gain is rarely used in digital force control applications.

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