Accurate Models of Planar Compliant Parallel Mechanisms

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Abstract: In order to model more precisely compliant mechanisms, each planar compliant joint is replaced by 3 virtual springs. An equation giving the static equilibrium of any compliant parallel mechanism (CPM) is then found and combined with the kinematic equations to formulate the Geometrico-Static Model (GSM) of planar CPM. Next, by differentiating this GSM, a Kinemato-Static Model is formulated, in which appears notably the compliant matrix of a CPM. At the end of the paper some findings about CPMs arising from the use of these models are given.

Keywords: compliant parallel mechanisms, modelling, stiffness matrix

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Modélisation exacte des mécanismes parallèlles compliants dans le plan

Résumé: Afin de modéliser plus exactement les mécanismes compliants, chaque articulation compliante est remplacée par 3 ressorts virtuels. Une équation décrivant l'équilibre statique de tout mécanisme parallèlle compliant (MPC) est donnée et combinée aux équations cinématiques afin de formuler un modèle géométrico-statique (MGS) des MPC plan. Ensuite, en différenciant ce MGS, un modèle cinémato-statique est formulé. Dans celui-ci apparaît notamment la matrice de compliance d'un MPC. À la fin du papier, sont donnés quelques enseignements sur ces MPCs tirés de l'utilisation de ces modèles.

Mots clés: mécanismes parallèles compliants, modélisation, matrice de raideur

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1 INTRODUCTION

The use of compliant joints as passive joints in parallel mechanisms (PM) eliminates clearance and wear, thereby improving the precision and the repeatability of the manipulator.
However, the behaviour of a compliant joint differs from that of a conventional joint and must be taken into account. The passive joint's stiffness is not zero in its main axis and is not infinite (perfectly stiff) in the other directions [1]. In this work, a compliant joint is not considered only as one joint but as a group of 3 virtual joints with 3 springs.
The additional virtual joints lead to considering a compliant parallel mechanism as a highly redondant mechanism. Thus some constraints must be satisfied, namely : geometric, kinematic (both corresponding to the closure of kinematic chains) and static constraints (corresponding to the static equilibrium between all the virtual springs).
Here, a Geometrico-Static Model (GSM) and a Kinemato-Static Model (KSM) are presented that enable the accurate computation of the position of a manipulator when the actuators' position and the external forces are known. Simulations with these models reveal some interesting characteristics that can be exploited to improve the design of a compliant parallel manipulator.

2 MODEL OF A CPM

2.1 Modelling of a joint

In order to calculate the motion due to the flexibilies of a compliant joint in the 3 directions, we should consider it not as a 1-DoF joint, but as a group of 3 virtual joints combined with 3 springs (Fig.1).
The stiffness in the main axis is called the primary stiffness and the stiffnesses in the other directions are called secondary stiffnesses. A ratio ρ between the secondary stiffnesses and the primary one indicates if

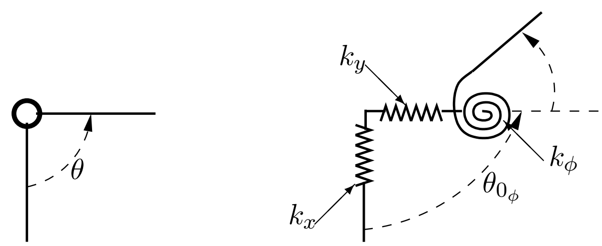

Figure 1: Conventional and realistic compliant revolute joint models

the behaviour of a compliant joint is close to that of a conventional joint (in which $\rho \rightarrow +\infty$).
In this work, it is assumed that the stiffnesses in a joint are not coupled. It is the case in cross section compliant joints [2]. Thus the matrix K_{joint} corresponding to only one isolated joint is a diagonal matrix composed by k_x, k_y and k_ϕ.

2.2 Constraints to be satisfied

2.2.1 Geometric constraints

The following equations are the constraints corresponding to the closure of loops formed by the legs of a PM:

$$\mathbf{x}_i = \mathbf{x}_j, \quad \forall (i,j) \tag{1}$$

where $\mathbf{x}_i = [x_i, y_i, \phi_i]^T$ is the pose vector of the end body of the i^{th} leg (respectively j).

2.2.2 Vector of free parameters

A vector of free parameters $\boldsymbol{\chi}$ can be defined such that $\boldsymbol{\theta}$, the complete articular vector of the mechanism, always satisfies the geometrical constraints :

$$\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\chi}) \tag{2}$$

The dimension l of this vector equals the number of degrees of freedom (DOF) of the kinematic equivalent mechanism.
We define $\mathbf{R} = \frac{d\boldsymbol{\theta}}{d\boldsymbol{\chi}}$, such that $d\boldsymbol{\theta} = \mathbf{R}d\boldsymbol{\chi}$.

2.2.3 Kinematic constraints

The global Jacobian matrix of the mechanism with n legs is defined as

$$\mathbf{J}_\theta = \frac{1}{n}\left[\mathbf{J}_{\theta_a}, \cdots, \mathbf{J}_{\theta_n}\right] = \frac{1}{n}\left[\frac{\partial \mathbf{x}_a}{\partial \boldsymbol{\theta}_a}, \cdots, \frac{\partial \mathbf{x}_n}{\partial \boldsymbol{\theta}_n}\right] \tag{3}$$

with $\mathbf{J}_{\theta_i}$ the Jacobian matrix and $\boldsymbol{\theta}_i$ the articular vector of the i^{th} leg. The relation between the effector's motion and the joints motion, substituting the geometric constraints, is

$$d\mathbf{x} = \mathbf{J}_\theta d\boldsymbol{\theta} = \mathbf{J}_\theta \mathbf{R} d\boldsymbol{\chi} \tag{4}$$

2.2.4 Static Equilibrium

The principle of virtual work gives the relation between the work done by the external force at the effector and the work done by the springs in each joint

$$(-\mathbf{t})^T d\boldsymbol{\theta} = \mathbf{f}^T d\mathbf{x} \tag{5}$$

where $\mathbf{f}$ is the vector of external efforts applied on the effector and $\mathbf{t}$ is the vector of forces and torques in the springs defined by

$$\mathbf{t} = -\mathbf{K}_\theta(\boldsymbol{\theta} - \boldsymbol{\theta}_0) \tag{6}$$

with $\mathbf{K}_\theta$ the joints stiffness matrix and $\boldsymbol{\theta}_0$ the vector of virtual springs' free lengths.
Then substituting all constraints in (5) and reducing by $(d\boldsymbol{\chi})$ —the vector of independent coordinates—, we obtain the static equilibrium of the CPM

$$\mathbf{R}^T\left[\mathbf{t} + \mathbf{J}_\theta^T \mathbf{f}\right] = \mathbf{R}^T \mathbf{s} = \mathbf{0} \tag{7}$$

We note $\mathbf{s}$ the vector representing at each joint the sum of the efforts due to springs and due to external efforts. The dimension of this vector $\mathbf{s}$ is m and the dimension of ($\mathbf{R}^T\mathbf{s}$) is l, so we can notice that some local sums (some coordinates s_k of $\mathbf{s}$) can differ from 0, even when the mechanism is in equilibrium, without any external effort.

2.3 Geometrico-Static Model

The solution of the GSM are the values of $\{\boldsymbol{\theta}, \mathbf{f}, \boldsymbol{\alpha}, \mathbf{x}\}$ that satisfy the geometric and the static constraints. In case of multiple possible solutions, some additional criteria like stability [3] should be used.
Moreover, since the components of $\boldsymbol{\theta}$ are internal parameters, it is theoretically possible to reduce the model of a planar CPM to a smaller system of 3 equations with 9 variables, namely

$$\mathbf{x} = \mathcal{G}(\boldsymbol{\alpha}, \mathbf{f}) \tag{8}$$

2.4 Kinemato-Static Model

This model aims at describing the relation between the motion of the effector, the motion of actuators and the variation of external efforts[3].
The GSM is first differentiated in order to obtain a relation under the following form

$$d\mathbf{x} = \mathbf{J}_{\theta_0} d\boldsymbol{\theta}_\mathbf{0} + \mathbf{C}_C d\mathbf{f} \tag{9}$$

The result is :

$$\begin{cases} \mathbf{J}_{\theta_0} = \mathbf{J}_\theta \mathbf{R}\, \mathbf{K}_\chi^{-1}\, \mathbf{R}^T \mathbf{K}_\theta \\ \\ \mathbf{C}_C = \mathbf{J}_\theta \mathbf{R}\, \mathbf{K}_\chi^{-1}\, \mathbf{R}^T \mathbf{J}_\theta^T \end{cases} \tag{10}$$

with $\mathbf{K}_\chi$ the stiffness matrix in the free parameter domain defined as

$$\mathbf{K}_\chi = \left[\mathbf{R}^T(\mathbf{K}_\theta - \mathbf{K}_G) - \mathbf{K}_R\right] \mathbf{R} \tag{11}$$

where $\mathbf{K}_G$ (See [4]) and $\mathbf{K}_R$ are defined as

$$\begin{cases} \mathbf{K}_G = \left[(\frac{\partial \mathbf{J}_\theta}{\partial \theta_1})^T \mathbf{f}, \cdots, (\frac{\partial \mathbf{J}_\theta}{\partial \theta_m})^T \mathbf{f}\right] \\ \\ \mathbf{K}_R = \left[(\frac{\partial \mathbf{R}}{\partial \theta_1})^T \mathbf{s}, \cdots, (\frac{\partial \mathbf{R}}{\partial \theta_m})^T \mathbf{s}\right] \end{cases} \tag{12}$$

Matrix $\mathbf{C}_C$ is the global compliance matrix of the mechanism. Matrix $\mathbf{J}_{\theta_0}$ is called the compliant Jacobian matrix.

3 Contributions of the model

By using the above models, we can accurately determine the motion of a CPM. The results —exactly equal to those given by MSC Adams when we use bushings to model compliant joints— show the differences in kinematic and static behaviour of a CPM and a conventional mechanism.

Kinematics

- The ratio ρ between the primary stiffness and the secondary stiffnesses of the joints has an impact on the kinematics of the mechanism. The larger ρ is, the closer the behaviour of the CPM will be to that of a conventional mechanism.
- When the compliant mechanism is close to the zero energy starting point, internal efforts and bendings are small, and hence its behaviour is closer to that of conventional mechanisms.
- The mechanisms built by replacing conventional joints by compliant joints do not have the same number of DoFs.

Statics

- This new KSM enables the computation of the effects of the mechanism's compliance.
- A mechanism in which the joint stiffnesses are high is less sensitive to the external efforts, but it requires more powerful actuators.

[3] The dynamic effects are not considered in this model.

CONCLUSION

The presented models better represent the nature of CPMs and hence enhance the accuracy of manipulators. Morevoer, the equations are simple and easy to solve. In the future, this model will be extended to spatial mechanisms and to compliant joints with coupled stiffness. The accuracy of the model will also be evaluated with a prototype.

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