A Procedure to Formulate Normal Impact of Planar Deformable Bodies as a Linear Complementarity Problem

Saeed Ebrahimi* and Peter Eberhard+

* Department of Mechanical Engineering, McGill University, Canada

+ Institute of Engineering and Computational Mechanics, University of Stuttgart, Germany
saeed.ebrahimi@mcgill.ca, eberhard@itm.uni-stuttgart.de

There are many applications dealing with contact/impact problems in industry and engineering and make their modeling an essential and very demanding topic in multibody dynamics. Impact events in multibody systems may arise from different sources. The most common type happens when moving bodies in the system collide. Clearances at different joints, mass capture and mass release, intermittent motion of bodies and others may be other sources of impact. Due to the nature of impact, high forces will be exerted to the impacting bodies during a very short period of time. Therefore, in order to describe the physical process correctly, considering local deformations becomes often an inevitable demand. In such situations, however, rigid body impact modeling can not be used anymore and instead, the impact problem of deformable bodies must be solved.

The formulation of impact kinematics in normal direction for planar deformable bodies can be started by obtaining a relation between normal gaps $\mathbf{g}_N$ of possible contact pairs and generalized coordinates $\mathbf{q}$ of bodies. Furthermore, a relation for the generalized coordinates in terms of the normal impact forces $\boldsymbol{\lambda}_N$ is to be obtained. Substitution of such a relation in the relation between $\mathbf{g}_N$ and $\mathbf{q}$ states the normal gaps vector $\mathbf{g}_N$ in terms of $\boldsymbol{\lambda}_N$. For this purpose, we consider the equations of motion as

$$\mathbf{M} \cdot \ddot{\mathbf{q}} = \mathbf{h} + \mathbf{W}_N \cdot \boldsymbol{\lambda}_N \ , \tag{1}$$

where $\mathbf{M}$ is the system mass matrix, $\ddot{\mathbf{q}}$ is the vector of generalized accelerations, $\mathbf{h}$ contains the generalized external, internal and Coriolis forces. The Lagrange multipliers vector $\boldsymbol{\lambda}_N$ denotes normal impact forces which are projected to the generalized directions through the matrix $\mathbf{W}_N$. Integrating $\ddot{\mathbf{q}}$ using the 4^{th} order Runge-Kutta integration approach yields a relation between the systems generalized coordinates $\mathbf{q}$ and the impact forces $\boldsymbol{\lambda}_N$. By successive evaluations and substitutions of the vector $\mathbf{q}$ for several stages of the 4^{th} order Runge-Kutta scheme one obtains, see [1],

$$\mathbf{q}_{i+1} = \underbrace{\frac{\Delta t^2}{6} \sum_{k=1}^{3} (\mathbf{M}_k^{-1} \cdot \mathbf{W}_{N_k})_i}_{\mathbf{W}_{q_i}} \cdot \boldsymbol{\lambda}_{N_i} + \underbrace{(\mathbf{q}_i + \Delta t \dot{\mathbf{q}}_i + \frac{\Delta t^2}{6} \sum_{k=1}^{3} (\mathbf{M}_k^{-1} \cdot \mathbf{h}_k)_i)}_{\mathbf{w}_{q_i}} \ , \tag{2}$$

where the index i is the i^{th} integration step, Δt is the step size and $\mathbf{M}_k^{-1}, \mathbf{W}_{N_k}$ and $\mathbf{h}_k$ $(k = 1, 2, 3)$ arise from equations of motion but are evaluated at the first three stages of the 4^{th} order Runge-Kutta method. The generalized coordinates can now be substituted from Eq. (2) into the relation of normal gaps to reach

$$\mathbf{g}_N = \mathbf{W}_{gq}^T \cdot \boldsymbol{\lambda}_N + \mathbf{w}_{gq} \qquad \text{with} \qquad \mathbf{g}_N \geq \mathbf{0} \ , \qquad \boldsymbol{\lambda}_N \geq \mathbf{0} \ , \qquad \mathbf{g}_N \cdot \boldsymbol{\lambda}_N = 0 \ , \tag{3}$$

where the complementarity relation between $\mathbf{g}_N$ and $\boldsymbol{\lambda}_N$ is utilized. This relation of normal impact will further be used to construct the linear complementarity problem (LCP), see [2], which is to be solved for calculation of impact forces $\boldsymbol{\lambda}_N$, see [1, 3]. As another possibility to formulate normal impact of planar deformable bodies, one may consider the complementarity relations on velocity level and attempt to find a relation between the velocity of normal gaps $\dot{\mathbf{g}}_N$ and $\boldsymbol{\lambda}_N$. In this case, we use the relation between $\dot{\mathbf{g}}_N$ and the generalized velocities $\dot{\mathbf{q}}$, see [3]. Again one can use the 4^{th} order Runge-Kutta integration approach but this time one finds a relation between the systems generalized velocities $\dot{\mathbf{q}}$ and $\boldsymbol{\lambda}_N$

$$\begin{aligned} \dot{\mathbf{q}}_{i+1} = & \underbrace{\left(\frac{\Delta t}{6} (\mathbf{M}_1^{-1} \cdot \mathbf{W}_{N_1} + \mathbf{M}_4^{-1} \cdot \mathbf{W}_{N_4})_i + \frac{\Delta t}{3} \sum_{k=2}^{3} (\mathbf{M}_k^{-1} \cdot \mathbf{W}_{N_k})_i \right)}_{\mathbf{W}_{v_i}} \cdot \boldsymbol{\lambda}_{N_i} + \\ & \underbrace{\left(\dot{\mathbf{q}}_i + \frac{\Delta t}{6} (\mathbf{M}_1^{-1} \cdot \mathbf{h}_1 + \mathbf{M}_4^{-1} \cdot \mathbf{h}_4)_i + \frac{\Delta t}{3} \sum_{k=2}^{3} (\mathbf{M}_k^{-1} \cdot \mathbf{h}_k)_i \right)}_{\mathbf{w}_{v_i}} = \mathbf{W}_{v_i} \cdot \boldsymbol{\lambda}_{N_i} + \mathbf{w}_{v_i} \ . \end{aligned} \tag{4}$$

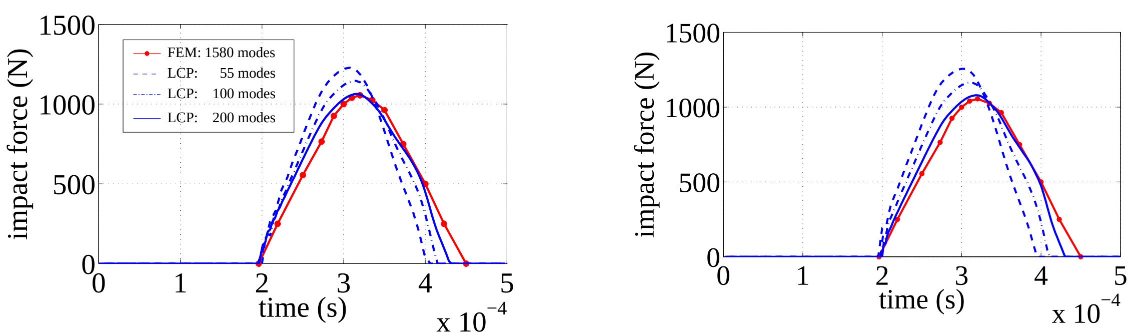
The generalized velocities can now be substituted from Eq. (4) into the relation of velocity of normal gaps to obtain

$$\dot{\mathbf{g}}_N = \mathbf{W}_{gv}^T \cdot \boldsymbol{\lambda}_N + \mathbf{w}_{gv} \qquad \text{with} \qquad \dot{\mathbf{g}}_N \geq \mathbf{0} \qquad \boldsymbol{\lambda}_N \geq \mathbf{0} \qquad \dot{\mathbf{g}}_N \cdot \boldsymbol{\lambda}_N = 0\ . \tag{5}$$

Similar to the formulation on position level, the formulation on velocity level may be followed to reach the required LCP. At this point, we make some remarks which have to be considered when implementing this approach:

- During the formulation of normal impact on position and velocity level, no coefficient of restitution is introduced for obtaining the impact law and the energy loss is taken into account directly by the damping in the material law of the deformable bodies. This is considered in the vector $\mathbf{h}$ of Eq. (1).
- Formulations presented on position and velocity level based on the explicit 4^{th} Runge-Kutta method are just used as a discretization method to obtain an analytical formulation between $\mathbf{g}_N$ and $\boldsymbol{\lambda}_N$ or between $\dot{\mathbf{g}}_N$ and $\boldsymbol{\lambda}_N$ and not for the main integration processes to proceed to the next time step.
- It is noticeable that the complementarity conditions used for impact analysis are also valid for continual contact of planar deformable bodies. In this way they will eliminate the necessity of switching between continual contact and impact formulations.
- The required LCP for normal impact on velocity level must be considered only for the corresponding active contact pairs, no matter whether friction is considered or not. The same holds for the case when normal impact on position level is used together with friction. However, normal impact on position level without friction may be considered for active and non-active contact pairs since the collision detection step may be done automatically as the result of the LCP on position level.

Some results of the frictionless impact simulation between two identical elastic disks are depicted in Fig. 1. In this example, the left disk has an approaching initial velocity of 0.5 m/s while the right disk is initially at rest. The quantities $E = 2e8\ Pa$, $\rho = 10\ kg/m^2$ and $\nu = 0.3$ for Young's modulus, density and Poisson's ratio are used, respectively. The results show that the formulations on both position and velocity level approach the precise results of FEM even for stiff planar deformable bodies provided that a proper number of eigenmodes of the FEM model for building the reduced model of deformable bodies together with an appropriate time step is chosen. Therefore, considering the material damping for treatment of energy loss in impact of planar deformable bodies is sufficient without having to introduce any coefficient of restitution.



a: position level   b: velocity level

Figure 1: Impact simulation of two identical elastic disks

References

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