THE LAGRANGIAN DERIVATION OF KANE'S EQUATIONS

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Abstract

The Lagrangian approach to the development of dynamics equations for a multi-body system, constrained or otherwise, requires solving the forward kinematics of the system at velocity level in order to derive the kinetic energy of the system. The kinetic-energy expression should then be differentiated multiple times to derive the equations of motion of the system. Among these differentiations, the partial derivative of kinetic energy with respect to the system generalized coordinates is specially cumbersome. In this paper, we will derive this partial derivative using a novel kinematic relation for the partial derivative of angular velocity with respect to the system generalized coordinates. It will be shown that, as a result of the use of this relation, the equations of motion of the system are directly derived in the form of Kane's equations.

Keywords: dynamics modelling, Kane's equations, multi-body system, virtual work

LA DÉRIVATION LAGRANGIENNE DES ÉQUATIONS DE KANE

Résumé

L'approche lagrangienne pour le développement des équations de dynamique pour un système multi-corps, contraint ou pas, exige la résolution de la cinématique directe du système au niveau des vitesses afin de determiner l'énergie cinétique du système. L'expression de l'énergie cinétique doit alors être dérivée plusieurs fois afin de detérminer les équations du mouvement du système. Parmi ces dérivations, la dérivée partielle de l'énergie cinétique par apport aux coordonnées généralisées du système est particulièrement complexe. Dans cet article, nous dériverons cette dérivée partielle en utilisant une relation cinématique novatrice pour la dérivée partielle de la vitesse angulaire par rapport aux coordonnées généralisées du système. Nous montrerons que, en raison de l'utilisation de cette relation, les équations du mouvement du système sont directement dérivées sous forme d'équations de Kane.

Mots clés: modélisation dynamique, équations de Kane, système multi-corps, travail virtuel

1 INTRODUCTION

Motivated by the need for the dynamics analysis of complex systems, many researchers have tried to develop the equations of motion of multi-body systems in novel forms more suitable for numerical computations, symbolic manipulations, or both. The Newton-Euler formalism, despite its strength, seemed unattractive because it requires the computation of all constraint forces and moments whereas energy-based methods, e.g., the Euler-Lagrange, disregard the constraint forces and moments because these forces and moments do not contribute to the total work performed. The Euler-Lagrange method, however, has its own issue that makes its application cumbersome. The issue is that it involves differentiating the kinetic and potential energies of the entire multi-body system with respect to the system generalized coordinates and velocities. These differentiations combined with the nonlinear nature of the relation between the twist of each body in the system to the system generalized coordinates render the "raw" application of the method to multi-body systems impractical.[2]

To simplify the above-mentioned differentiations of the kinetic energy, we derive closed-form expressions for the partial derivatives of the translational and angular velocities of a body with respect to the system generalized coordinates and velocities. Among these relations, it is the partial derivative of the angular velocity with respect to the generalized coordinates that is new. This relation was derived a few years ago [2] for the purpose of obtaining the linearized kinematics of structurally flexible serial manipulators. The relation had the condition that the generalized coordinates of the system be *independent*. Moreover, the application of that relation to the derivation of dynamics equations was not discussed. In this paper, we use the above-mentioned expressions to differentiate the kinetic energy of an unconstrained system. Furthermore, we will extend that relation to the case of constrained systems and use the result to derive the equations of motion of constrained multi-body systems.

It will be demonstrated that, using the above-mentioned closed-form relations, the Euler-Lagrange formalism results in a set of dynamics equations which are known as Kane's equations. Among all attempts to develop more efficient formulations for the equations of motion of multi-body systems Kane's equations [3; 4] have proven to be probably, by far, the most controversial. The controversy, however, is not on the technical merits or the accuracy of Kane's final results; rather, it mostly concerns the originality of the equations, and the way they are obtained.[3] Kane's equations have been compared with the earlier results of Gibbs and Appell [6], Jourdain [7] and Maggi [8]. For a brief summary of such discussions, see [9].

In Section 2, we will briefly discuss the principle of virtual work and show how the principle can be used to derive the dynamics equations for constrained and unconstrained multi-body systems. Section 3 is dedicated to the derivation of equations of motion for systems with tree structures. Section 4 discusses the same for constrained systems, the paper concluding with some remarks in Section 5.

2 THE PRINCIPLE OF VIRTUAL WORK

The D'Alembert principle of virtual work in Lagrange's form [10; 11], or as Papastavridis [11] calls it the Lagrange Principle, can be reformulated to obtain

$$\boldsymbol{\phi}^T \delta \mathbf{q} = 0 \quad \text{where} \quad \boldsymbol{\phi} \triangleq \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} - \mathbf{f} \tag{1}$$

where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the vectors of generalized coordinates and velocities, respectively, and T denotes the system kinetic energy. The vector $\mathbf{f}$ of total *impressed* generalized forces in the above equation represents the active forces applied on the multi-body system. This force can be computed from the definition of virtual work:

$$\delta W = \sum_{i=1}^{n} \left(\mathbf{f}_i^T \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}} + \mathbf{n}_i^T \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}}\right) \delta \mathbf{q} \equiv \mathbf{f}^T \delta \mathbf{q} \tag{2}$$

where $\mathbf{f}_i$ and $\mathbf{n}_i$ are the resultant impressed force and moment applied on the ith body, respectively; $\mathbf{r}_i$ is the position vector of the point at which $\mathbf{f}_i$ is applied, and $\boldsymbol{\omega}_i$ is the angular velocity of the ith body of the system.

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[2] For a three-link planar mechanism with only three degrees of freedom, for example, one has to write pages of equations to be able to properly compute the kinetic energy and differentiate it as needed [1].

[3] A fundamental issue is that Kane considers the concept of virtual displacement "objectionable" [5]. Even though this difference of opinion is of major importance conceptually and affects how Kane's equations should be derived, i.e., based on what principle of mechanics, it has no bearing on the equations derived.

When $\mathbf{q}$ is an n-dimensional vector of independent generalized coordinates of the system, all entries of $\delta\mathbf{q}$, i.e., the virtual changes in the coordinates, will be independent and arbitrary. The Lagrange equation is then derived in vector form as

$$\boldsymbol{\phi} = \mathbf{0} \tag{3}$$

and the vector of generalized forces can be obtained as

$$\mathbf{f} = \sum_{i=1}^{n} \Big[\big(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}}\big)^T \mathbf{f}_i + \big(\frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}}\big)^T \mathbf{n}_i \Big] \tag{4}$$

However, if the generalized coordinates are not independent, the virtual changes cannot be imposed independent of each other, and the equations of motion will not be as simple.

In the absolute majority of applications, the constraints can be written in the Pfaffian differential form as

$$\mathbf{A}(\mathbf{q}, t)\, d\mathbf{q} = \mathbf{b}(\mathbf{q}, t)\, dt \tag{5}$$

where $\mathbf{A}$ is assumed to be an $r \times n$-dimensional, full-rank matrix. Equivalently, the above constraint equation can be written as

$$\mathbf{A}(\mathbf{q}, t)\, \dot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, t) \tag{6}$$

Any vector that, when replacing $\dot{\mathbf{q}}$, satisfies the above equation is termed an *admissible velocity*.

Virtual displacements have two properties: They can be imposed instantly, i.e., in frozen time, and they must comply with the system constraints. The former property is by definition, and the latter is due to the principle [9]. Consequently, the virtual changes of the generalized coordinates should satisfy eq. (5) in the form below:

$$\mathbf{A}(\mathbf{q}, t)\, \delta\mathbf{q} = \mathbf{0} \tag{7}$$

Geometrically, this equation means that the virtual change must lie in the null-space of the matrix $\mathbf{A}$, i.e., $\delta\mathbf{q}$ must be perpendicular to all rows of $\mathbf{A}$. Hence, the dimension of the subspace within which $\delta\mathbf{q}$ can arbitrarily vary is $m \triangleq n - r$. On the other hand, according to the Lagrange Principle (1), $\delta\mathbf{q}$ must also be perpendicular to the n-dimensional vector $\boldsymbol{\phi}$. This simply means that $\boldsymbol{\phi}$ has to lie in the subspace spanned by the rows of $\mathbf{A}$, namely, its *row space*.

Because the system generalized coordinates and velocities are constrained by r independent constraint equations, only m of the generalized velocities will be independent. As such, we should be able to find an m-dimensional vector $\mathbf{u}$—with independent entries—which can produce all admissible velocities $\dot{\mathbf{q}}$ through the $n \times m$-dimensional, full-rank matrix $\mathbf{B}(\mathbf{q}, t)$[4] and the n-dimensional vector $\mathbf{d}(\mathbf{q}, t)$ through

$$\dot{\mathbf{q}} = \mathbf{B}(\mathbf{q}, t)\, \mathbf{u} + \mathbf{d}(\mathbf{q}, t) \quad \text{with} \quad \mathbf{B}(\mathbf{q}, t) \triangleq \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}} \tag{8}$$

The entries of $\mathbf{u}$ may or may not be time-derivatives themselves; those entries that belong to the latter category are called quasi-velocities[5], while the others are simply generalized velocities.

Since $\dot{\mathbf{q}}$ has to be admissible, it should satisfy the constraint equation. Therefore, we should have

$$\mathbf{A}(\mathbf{q}, t)\, \mathbf{B}(\mathbf{q}, t)\, \mathbf{u} + \mathbf{A}(\mathbf{q}, t)\, \mathbf{d}(\mathbf{q}, t) - \mathbf{b}(\mathbf{q}, t) = \mathbf{0}$$

for any given $\mathbf{u}$. Consequently, we obtain

$$\mathbf{A}(\mathbf{q}, t)\, \mathbf{B}(\mathbf{q}, t) = \mathbf{O}_{r \times m} \quad \text{and} \quad \mathbf{A}(\mathbf{q}, t)\, \mathbf{d}(\mathbf{q}, t) = \mathbf{b}(\mathbf{q}, t) \tag{9}$$

The first equation shows that the range space of $\mathbf{B}$ is orthogonal to all vectors in the row space of $\mathbf{A}$; as explained above, $\boldsymbol{\phi}$ is one such vector. Thus, we can conclude that

$$\boldsymbol{\phi}^T \mathbf{B} = \mathbf{0}^T \qquad \text{or} \qquad \mathbf{B}^T \boldsymbol{\phi} = \mathbf{0}$$

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[4] The rows of $\mathbf{B}$ are, in fact, what Kane [4] calls *partial velocities*.

[5] These are known as *generalized speeds* in Kane's formulation [4]. Terms quasi-coordinates and quasi-velocities, however, go back to the beginning of the 20th century [12].

Hence, if we can find a complete set of independent generalized velocities, quasi-velocities, or both, such as the entries of $\mathbf{u}$, then the Lagrange equations for the system can be written as

$$\big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \boldsymbol{\phi} = \mathbf{0} \tag{10}$$

The above m equations are independent from each other and fully express the dynamics of the system.

Similarly, we can derive the relation below for the generalized forces:

$$\big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \mathbf{f} = \big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \sum_{i=1}^{n} \Big(\big(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}}\big)^T \mathbf{f}_i + \big(\frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}}\big)^T \mathbf{n}_i\Big) \tag{11}$$

3 DYNAMICS OF MULTI-BODIES WITH A TREE STRUCTURE

In this case, one can readily choose the joint values as a set of independent generalized coordinates. Therefore, the Lagrange equation can be written as

$$\frac{d}{dt}\big(\frac{\partial T}{\partial \dot{\mathbf{q}}}\big) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{f}_{\text{nc}} \tag{12}$$

where the entries of $\mathbf{q}$ are the independent generalized coordinates of the system, and those of $\mathbf{f}_{\text{nc}}$ are the nonconservative generalized forces acting on the multi-body system. Scalar function V represents the system potential energy.

3.1 The kinetic energy

The kinetic energy of the system of n rigid-bodies can be computed as

$$T = \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{\omega}_i^T \mathbf{I}_i\, \boldsymbol{\omega}_i + \frac{1}{2}\sum_{i=1}^{n} \mathbf{v}_i^T m_i \mathbf{v}_i \tag{13}$$

where $\boldsymbol{\omega}_i$ and $\mathbf{I}_i$ respectively are the angular velocity and the centroidal moment of inertia of the body, both expressed in the ith body frame; $\mathbf{v}_i$ and m_i are the velocity of the body mass-centre and the mass of the body, respectively. The mass-centre velocity is expressed in the inertial frame.

The angular and the translational velocities are, in general, nonlinear functions of $\mathbf{q}$ the generalized coordinates of the system and linear functions of $\dot{\mathbf{q}}$ the generalized velocities. These functional relations can be established through the forward kinematics of the system. They will be of the form

$$\boldsymbol{\omega}_i \equiv \boldsymbol{\omega}_i(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{J}_{\omega i}\, \dot{\mathbf{q}}, \qquad \Rightarrow \qquad \dot{\boldsymbol{\omega}}_i = \frac{d}{dt}\big(\mathbf{J}_{\omega i}\, \dot{\mathbf{q}}\big) = \mathbf{J}_{\omega i}\, \ddot{\mathbf{q}} + \dot{\mathbf{J}}_{\omega i}\, \dot{\mathbf{q}} \tag{14}$$

$$\mathbf{v}_i \equiv \mathbf{v}_i(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{J}_{vi}\, \dot{\mathbf{q}} \qquad \Rightarrow \qquad \dot{\mathbf{v}}_i = \frac{d}{dt}\big(\mathbf{J}_{vi}\, \dot{\mathbf{q}}\big) = \mathbf{J}_{vi}\, \ddot{\mathbf{q}} + \dot{\mathbf{J}}_{vi}\, \dot{\mathbf{q}} \tag{15}$$

where, for simiplicity, we have assumed that the system is sceleronomic, and $\mathbf{J}_{\omega i}$ and $\mathbf{J}_{vi}$ are defined as

$$\mathbf{J}_{\omega i} \triangleq \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}} \quad \text{and} \quad \mathbf{J}_{vi} \triangleq \frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}} \tag{16}$$

Traditionally, at this stage, the kinetic-energy expression is expanded in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$ by using eqs. (14) and (15), and the expression is reformulated as[6]

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

Thereafter, one has to work with the generalized inertia matrix $\mathbf{M}$ of the system, and differentiate the entries of that matrix with respect to $\mathbf{q}$. Here, however, we will try to continue with the kinetic energy as expressed in eq. (13). To that end, we recall the following theorem [2]. A new proof of the theorem is produced in Appendix A.

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[6] For example, one can refer to [13; 14; 1], among others.

Theorem 1. *The partial derivative of the angular velocity $\boldsymbol{\omega}$ of a rigid body in a kinematic chain with respect to the chain independent generalized-coordinate vector* $\mathbf{q}$ *can be expressed in terms of the Jacobian $\mathbf{J}_\omega$, its time rate, and the cross-product matrix $[\boldsymbol{\omega}\times]$ of the angular velocity as*

$$\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{q}} = \dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega \tag{17}$$

Another relation that we will need in our derivation is the partial derivative of the velocity of the body mass-centre with respect to the system generalized coordinates:

$$\begin{aligned} & \frac{\partial \mathbf{v}_i}{\partial \mathbf{q}} = \frac{\partial \dot{\mathbf{c}}_i}{\partial \mathbf{q}} = \frac{d}{dt}\Big(\frac{\partial \mathbf{c}_i}{\partial \mathbf{q}}\Big) = \frac{d}{dt}\Big(\frac{\partial \dot{\mathbf{c}}_i}{\partial \dot{\mathbf{q}}}\Big) = \frac{d}{dt}\Big(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}}\Big) \\ \Rightarrow \quad & \frac{\partial \mathbf{v}_i}{\partial \mathbf{q}} = \dot{\mathbf{J}}_{vi} \end{aligned} \tag{18}$$

In the above equations, vector $\mathbf{c}_i$ denotes the position vector of the centre of mass of the ith body expressed in the fixed frame.

Below, we will use eqs. (17) and (18) to derive all the terms of the Lagrange equation (12).

Computing $\frac{d}{dt}(\partial T/\partial \dot{\mathbf{q}})$

Using the relations derived above, We can compute the partial derivative of T with respect to the generalized velocities $\dot{\mathbf{q}}$ as

$$\begin{aligned} \frac{\partial T}{\partial \dot{\mathbf{q}}} &= \sum_{i=1}^{n}\Big(\frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}}\Big)^T \mathbf{I}_i\,\boldsymbol{\omega}_i + \sum_{i=1}^{n}\Big(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}}\Big)^T m_i \mathbf{v}_i \\ &= \sum_{i=1}^{n} \mathbf{J}_{\omega i}^T\,\mathbf{I}_i\,\mathbf{J}_{\omega i}\,\dot{\mathbf{q}} + \sum_{i=1}^{n}\mathbf{J}_{vi}^T\,m_i\,\mathbf{J}_{vi}\,\dot{\mathbf{q}} \end{aligned} \tag{19}$$

The right-hand side of eq. (19) can then be written in a more compact form:

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{M}_i \mathbf{J}_i\,\dot{\mathbf{q}} \tag{20}$$

In the above equation, the Jacobian matrix $\mathbf{J}_i$ and the inertia dyad $\mathbf{M}_i$ of the ith link are defined as

$$\mathbf{J}_i \triangleq \begin{bmatrix} \mathbf{J}_{\omega i} \\ \mathbf{J}_{vi} \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_i \triangleq \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix} \tag{21}$$

It should be stressed that, as defined above, the two blocks $\mathbf{J}_{\omega i}$ and $\mathbf{J}_{vi}$ of $\mathbf{J}_i$ are referred to two different coordinate frames, namely, the body frame and the base frame, respectively.

Finally, differentiating eq. (20) with respect to time, we have

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\mathbf{q}}}\Big) = \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{M}_i \frac{d}{dt}(\mathbf{J}_i \dot{\mathbf{q}}) + \sum_{i=1}^{n} \dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}} \tag{22}$$

Computing $\partial T/\partial \mathbf{q}$

Equations (17) and (18) can be used to differentiate the kinetic-energy expression (13) with respect to the generalized coordinates:

$$\begin{aligned} \frac{\partial T}{\partial \mathbf{q}} &= \sum_{i=1}^{n}\Big(\frac{\partial \boldsymbol{\omega}_i}{\partial \mathbf{q}}\Big)^T \mathbf{I}_i\,\boldsymbol{\omega}_i + \sum_{i=1}^{n}\Big(\frac{\partial \mathbf{v}_i}{\partial \mathbf{q}}\Big)^T m_i \mathbf{v}_i \\ &= \sum_{i=1}^{n}\big(\dot{\mathbf{J}}_{\omega i} + [\boldsymbol{\omega}_i\times]\,\mathbf{J}_{\omega i}\big)^T\,\mathbf{I}_i\,\mathbf{J}_{\omega i}\,\dot{\mathbf{q}} + \sum_{i=1}^{n}\dot{\mathbf{J}}_{vi}^T\,m_i\,\mathbf{J}_{vi}\,\dot{\mathbf{q}} \end{aligned} \tag{23}$$

One should notice that $\mathbf{I}_i$ is expressed in a body-attached frame; as such, it is not affected by the motion of the manipulator. The above result can be rewritten in the more compact form below:

$$\frac{\partial T}{\partial \mathbf{q}} = \sum_{i=1}^{n} \left(\dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i - \mathbf{J}_i^T \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \right) \dot{\mathbf{q}} \tag{24}$$

in which the skew-symmetric, 6×6 angular-velocity dyad $\mathbf{W}_i$ is defined as

$$\mathbf{W}_i \triangleq \begin{bmatrix} [\boldsymbol{\omega}_i \times] & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{O}_{3\times3} \end{bmatrix} \tag{25}$$

3.2 The potential energy

The effect of conservative forces, e.g., gravity, can be properly accounted for using a potential energy function. For brevity, we only consider the gravitational potential energy here; if there are flexibilities in the system, the potential energy should be complemented with the elastic potential energy. The potential energy can therefore be expressed as

$$V = -\sum_{i=1}^{n} m_i \mathbf{c}_i^T \mathbf{g} \tag{26}$$

where $\mathbf{g}$ is the gravitational acceleration. Therefore, the partial derivative of the potential energy with respect to the generalized coordinates can be computed as

$$\frac{\partial V}{\partial \mathbf{q}} = -\sum_{i=1}^{n} m_i \Big(\frac{\partial \mathbf{c}_i}{\partial \mathbf{q}}\Big)^T \mathbf{g} = -\sum_{i=1}^{n} m_i \Big(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}}\Big)^T \mathbf{g}$$

Then, the vector of conservative generalized forces can be obtained from

$$\mathbf{f}_{\mathrm{c}} \triangleq -\frac{\partial V}{\partial \mathbf{q}} = \sum_{i=1}^{n} m_i \mathbf{J}_{vi}^T \mathbf{g} \tag{27}$$

Of course, one can also include gravity in the model by propagating it from the base of the chain upward, which amounts to the base having an acceleration of $-\mathbf{g}$. For one such algorithm, one can refer to [15].

For large manipulators deployed on orbit, the vector of gravitational acceleration may vary from element to element due to the dependence of the gravitational acceleration on the distance of the link centre of mass from the Earth centre. If one or more of the bodies is so big that its centres of mass and gravity do not coincide, an extra term due to the effect of the resulting gravitational moment should also be added to the expression of the potential energy. In such a case, the weight of the body applies a moment known as the *gravity-gradient torque* [16] about the centre of mass of the body. These gravitational effects cannot be handled by the propagation method mentioned above because, in this case, the gravitational acceleration is a vector function of the system generalized coordinates, not just a constant vector.

3.3 Applied forces and moments

Let us assume that, in addition to the joint-actuation forces and moments, represented here by $\boldsymbol{\tau}$, there are impressed external forces $\mathbf{f}_i^{\mathrm{ex}}$ and moments $\mathbf{n}_i^{\mathrm{ex}}$ applied on the bodies; these external forces are assumed to be applied at the mass centres. We further assume that both $\mathbf{f}_i^{\mathrm{ex}}$ and $\mathbf{n}_i^{\mathrm{ex}}$ are expressed in the body frame of the ith body. Then, the generalized force $\mathbf{f}_{\mathrm{nc}}$ applied on the system can be computed from

$$\mathbf{f}_{\mathrm{nc}} = \boldsymbol{\tau} + \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{w}_i^{\mathrm{ex}} \qquad \text{where} \qquad \mathbf{w}_i^{\mathrm{ex}} \triangleq \begin{bmatrix} \mathbf{1} & \mathbf{O}_{3\times3} \\ \mathbf{O}_{3\times3} & \mathbf{R}_i \end{bmatrix} \begin{bmatrix} \mathbf{n}_i^{\mathrm{ex}} \\ \mathbf{f}_i^{\mathrm{ex}} \end{bmatrix} \tag{28}$$

Rotation matrix $\mathbf{R}_i$ represents the rotation from the base-frame to the body frame of the ith link.

If there is damping in the joints, a generalized damping force $\mathbf{f}_{\mathrm{d}}$ will be added to the right-hand side of the expression of $\mathbf{f}_{\mathrm{nc}}$ in eq. (28).

3.4 The Lagrange equations

Substituting eqs. (22) and (23) in the left-hand side of Lagrange's equation (12), we obtain

$$\sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{M}_i \frac{d}{dt}\big(\mathbf{J}_i \dot{\mathbf{q}}\big) + \sum_{i=1}^{n} \dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}} - \sum_{i=1}^{n} \big(\dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i - \mathbf{J}_i^T \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i\big)\, \dot{\mathbf{q}} = \mathbf{f}_{\mathrm{nc}} + \mathbf{f}_{\mathrm{c}} \tag{29}$$

$$\Rightarrow \quad \sum_{i=1}^{n} \mathbf{J}_i^T \Big[\mathbf{M}_i \frac{d}{dt}\big(\mathbf{J}_i \dot{\mathbf{q}}\big) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}}\Big] = \mathbf{f} \tag{30}$$

where $\mathbf{f}$ is the sum of conservative and nonconservative impressed generalized forces. From eq. (4) and the definition of $\mathbf{J}_i$ in eq. (21), we can readily see that

$$\mathbf{f} = \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{w}_i \quad \text{where} \quad \mathbf{w}_i \triangleq \begin{bmatrix} \mathbf{1} & \mathbf{O}_{3\times 3} \\ \mathbf{O}_{3\times 3} & \mathbf{R}_i \end{bmatrix} \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{31}$$

Therefore, the dynamics equations can be written as

$$\sum_{i=1}^{n} \mathbf{J}_i^T \Big[\mathbf{M}_i \frac{d}{dt}\big(\mathbf{J}_i \dot{\mathbf{q}}\big) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}}\Big] = \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{w}_i \tag{32}$$

The part of the above equation within the brackets includes the inertial parts of the Newton and the Euler equations because

$$\mathbf{M}_i \frac{d}{dt}\big(\mathbf{J}_i \dot{\mathbf{q}}\big) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}} = \begin{bmatrix} \mathbf{I}_i\, \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i\, \boldsymbol{\omega}_i \\ m_i \dot{\mathbf{v}}_i \end{bmatrix} \tag{33}$$

which means eq. (32), in effect, is the set of Kane's equations for the multi-body system.

On the other hand, eq. (30) can be rewritten as

$$\sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{M}_i \mathbf{J}_i \ddot{\mathbf{q}} + \sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i\big)\, \dot{\mathbf{q}} = \mathbf{f}$$

which can be simplified to

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}_{\mathrm{nc}} + \mathbf{f}_{\mathrm{c}} \tag{34}$$

$$\mathbf{M} \triangleq \sum_{i=1}^{n} \mathbf{J}_i^T \mathbf{M}_i \mathbf{J}_i \quad \text{and} \quad \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \Big[\sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i\big)\Big]\, \dot{\mathbf{q}} \tag{35}$$

The positive-definite, symmetric matrix $\mathbf{M}$ is the system generalized inertia matrix. If taken to the other side of the equation, vector $\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$ will represent the vector of centrifugal and Coriolis generalized forces.

Let us define a matrix $\mathbf{C}$ as a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$ as

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \triangleq \sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i + \mathbf{M}_i \mathbf{W}_i \mathbf{J}_i\big) \tag{36}$$

It can readily be seen that, because $\mathbf{J}_i \dot{\mathbf{q}}$ lies in the null-space of $\mathbf{W}_i$,

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \equiv \sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i + \mathbf{M}_i \mathbf{W}_i \mathbf{J}_i\big)\, \dot{\mathbf{q}} \;\equiv \sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i\big)\, \dot{\mathbf{q}} \equiv \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) \tag{37}$$

Hence, eq. (34) can be reformulated in the form below:

$$\mathbf{M}\, \ddot{\mathbf{q}} + \mathbf{C}\, \dot{\mathbf{q}} = \mathbf{f}_{\mathrm{nc}} + \mathbf{f}_{\mathrm{c}} \tag{38}$$

Matrix $\mathbf{C}$, as defined in eq. (36), has the interesting property that $\dot{\mathbf{M}} - 2\mathbf{C}$ is skew-symmetric. For verification, we notice that

$$\begin{aligned}\dot{\mathbf{M}} - 2\mathbf{C} &= \sum_{i=1}^{n} \big(\dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i + \mathbf{J}_i^T \mathbf{M}_i \dot{\mathbf{J}}_i\big) - 2\sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{M}_i \dot{\mathbf{J}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i + \mathbf{M}_i \mathbf{W}_i \mathbf{J}_i\big) \\ &= \Big[\sum_{i=1}^{n} \big(\dot{\mathbf{J}}_i^T \mathbf{M}_i \mathbf{J}_i - \mathbf{J}_i^T \mathbf{M}_i \dot{\mathbf{J}}_i\big)\Big] - 2\Big[\sum_{i=1}^{n} \mathbf{J}_i^T \big(\mathbf{W}_i \mathbf{M}_i \mathbf{J}_i + \mathbf{M}_i \mathbf{W}_i \mathbf{J}_i\big)\Big]\end{aligned}$$

which is the sum of two skew-symmetric matrices, thus being skew-symmetric itself. It should be noted that the number of floating-point operations required for solving eq. (38) is more than that for eq. (34); however, the former equation has application in proving the stability of different robot control schemes; see [13; 17; 18] for some examples.

4 DYNAMICS OF CONSTRAINED MULTI-BODIES

As we saw in Section 2, from eqs. (3) and (10), there is one essential difference between the dynamics equations of unconstrained and constrained systems: In the latter, it is not the vector $\boldsymbol{\phi}$ which is vanishing; rather, it is its orthogonal projection along $\partial\dot{\mathbf{q}}/\partial u_i$ directions for $i = 1, \cdots, m$:

$$\big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \Big[\frac{d}{dt}\big(\frac{\partial T}{\partial \dot{\mathbf{q}}}\big) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}}\Big] = \big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \mathbf{f}_{\mathrm{nc}} \tag{39}$$

Reviewing the developments reported in Section 3, one realizes that we have used the assumption of independent generalized coordinates in two locations: the differentiation of kinetic energy with respect to the generalized coordinates—where we used Theorem 1—and the derivation of the generalized forces, both conservative and nonconservative. From these two, the latter has already been addressed through equation (11). The former, however, will be discussed below.

For our purposes, the computation of $\partial T/\partial \mathbf{q}$ hinged on Theorem 1, which provides the partial derivative of the angular velocity of a body within the kinematic chain with respect to the chain independent generalized coordinates. In case where the generalized coordinates are not independent, we can use the following result:

Theorem 2. *The variation of the angular velocity $\boldsymbol{\omega}$ of a rigid body in a kinematic chain due to a virtual change $\delta\mathbf{q}$ in the chain generalized coordinates can be expressed in terms of the Jacobian $\mathbf{J}_\omega$, its time rate, and the cross-product matrix $[\boldsymbol{\omega}\times]$ of the angular velocity as*

$$\big(\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{q}}\big)\delta\mathbf{q} = \big(\dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega\big)\delta\mathbf{q} \tag{40}$$

For the proof, one can refer to Appendix A.

Hence, if the generalized coordinates are subject to the Pfaffian constraints of eq. (5), then we can invoke a reasoning similar to that used in Section 2 to show that—for an independent, complete set $\mathbf{u}$ of generalized velocities, quasi-velocities, or both—we will have

$$\big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \big(\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{q}}\big) = \big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \big(\dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega\big) \tag{41}$$

Substituting eqs. (41) and (22) in eq. (39), we will obtain

$$\big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \sum_{i=1}^{n} \mathbf{J}_i^T \Big[\mathbf{M}_i \frac{d}{dt}(\mathbf{J}_i \dot{\mathbf{q}}) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}}\Big] = \big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \mathbf{f} \tag{42}$$

We can rewrite the left-hand side of the above equation as

$$\begin{aligned}\mathrm{LHS} &= \sum_{i=1}^{n} \big(\mathbf{J}_i \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\big)^T \Big[\mathbf{M}_i \frac{d}{dt}(\mathbf{J}_i \dot{\mathbf{q}}) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}}\Big] \\ &= \sum_{i=1}^{n} \mathbf{T}_i^T \Big[\mathbf{M}_i \frac{d}{dt}(\mathbf{J}_i \dot{\mathbf{q}}) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}}\Big] \end{aligned} \tag{43}$$

where matrix $\mathbf{T}_i$ is defined as

$$\mathbf{T}_i \triangleq \mathbf{J}_i \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}} \equiv \begin{bmatrix} \dfrac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}} \dfrac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}} \\ \dfrac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}} \dfrac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}} \end{bmatrix} \equiv \begin{bmatrix} \dfrac{\partial \boldsymbol{\omega}_i}{\partial \mathbf{u}} \\ \dfrac{\partial \mathbf{v}_i}{\partial \mathbf{u}} \end{bmatrix} \tag{44}$$

On the other hand, the right-hand side of eq. (42) can be rewritten as

$$\text{RHS} = \Big(\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}}\Big)^T \sum_{i=1}^{n} \Big(\Big(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\mathbf{q}}}\Big)^T \mathbf{f}_i + \Big(\frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}}\Big)^T \mathbf{n}_i \Big) = \sum_{i=1}^{n} \Big(\mathbf{J}_i \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{u}} \Big)^T \mathbf{w}_i = \sum_{i=1}^{n} \mathbf{T}_i^T \mathbf{w}_i \tag{45}$$

Therefore, the dynamics equations can be written as

$$\sum_{i=1}^{n} \mathbf{T}_i^T \Big[\mathbf{M}_i \frac{d}{dt}(\mathbf{J}_i \dot{\mathbf{q}}) + \mathbf{W}_i \mathbf{M}_i \mathbf{J}_i \dot{\mathbf{q}} \Big] = \sum_{i=1}^{n} \mathbf{T}_i^T \mathbf{w}_i \tag{46}$$

As seen from eq. (44), $\mathbf{T}_i$ is a matrix composed of the partial derivatives of the angular and translational velocities[7] of the ith body with respect to the set of independent velocities $\mathbf{u}$. Hence, the equation is in the form known as Kane's equations.

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5 CONCLUSIONS

Dynamics modelling of both constrained and unconstrained multi-body systems using the Euler-Lagrange approach was discussed in this paper. The approach requires differentiating the kinetic energy of the system with respect to the system generalized coordinates and velocity and subsequently differentiating the latter with respect to time. In the literature, the partial derivative of kinetic energy with respect to the generalized coordinates has been related to the partial derivative of the elements of the mass matrix with respect to the same variables. Due to the complicated relation between the elements of the mass matrix and these coordinates, the closed-form derivation of the dynamics model of a multi-body system through Lagrangian approach would stop at this point.

In this paper, we derived all the relevant partial derivatives in closed form. The partial derivative of particular interest was that of the kinetic energy with respect to the system generalized coordinates, particularly the part of the kinetic energy pertaining to the rotational motion. Written in body-attached frames, the link inertia tensors become constants, thus leaving the body angular velocities as the only variables. As such, we derived the partial derivative of the angular velocity with respect to the system generalized coordinates, in both cases of unconstrained and constrained systems, in closed form. Subsequently, the dynamics equations of the system were derived; the result was the equations of motion of the system in the form of Kane's equations.

APPENDIX A

Theorem 1. *The partial derivative of the angular velocity* $\boldsymbol{\omega}$ *of a rigid body in a kinematic chain with respect to the chain independent generalized-coordinate vector* $\mathbf{q}$ *can be expressed in terms of the Jacobian* $\mathbf{J}_\omega$*, its time rate, and the cross-product matrix* $[\boldsymbol{\omega}\times]$ *of the angular velocity as*

$$\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{q}} = \dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega \tag{A-1}$$

Proof: The angular velocity of a body can be expressed as a function of the body Euler parameters $\boldsymbol{\eta}$ and their time derivatives, i.e., as $\boldsymbol{\omega} \equiv \boldsymbol{\omega}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$. More specifically, as seen from eq. (B-19) in the body frame, we have

$$\tilde{\boldsymbol{\omega}} = 2\dot{\boldsymbol{\eta}} \otimes \boldsymbol{\eta}^* = 2\boldsymbol{\eta}^* \circledast \dot{\boldsymbol{\eta}} \tag{A-2}$$

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[7] Notice that the angular velocity and the translational velocity are expressed in two different frames, the former in a body-attached frame and the latter in an inertial frame. As such, the array containing $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ is not technically the *twist*.

where $\tilde{\boldsymbol{\omega}} \triangleq [\boldsymbol{\omega}^T \; 0]^T$ is the augmented angular-velocity vector, and $\boldsymbol{\eta}^*$ is the conjugate of $\boldsymbol{\eta}$, as defined in item 3 of Appendix B. Here, we have used the properties of the quaternion composition operators $\circledast$ and $\otimes$. The definitions of these operators along with some of their properties are given in Appendix B.

Differentiating eq. (A-2) with respect to the generalized coordinates $\mathbf{q}$, we can use the chain rule to obtain

$$\begin{aligned}\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\mathbf{q}} &= \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\boldsymbol{\eta}}\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}} + \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\boldsymbol{\eta}}}\frac{\partial\dot{\boldsymbol{\eta}}}{\partial\mathbf{q}} = 2\boldsymbol{\eta}^* \circledast \frac{\partial\dot{\boldsymbol{\eta}}}{\partial\mathbf{q}} + 2\dot{\boldsymbol{\eta}} \otimes \frac{\partial\boldsymbol{\eta}^*}{\partial\mathbf{q}} \\ &= 2\boldsymbol{\eta}^* \circledast \frac{d}{dt}\big(\frac{\partial\dot{\boldsymbol{\eta}}}{\partial\dot{\mathbf{q}}}\big) - 2\dot{\boldsymbol{\eta}} \otimes \boldsymbol{\eta}^* \otimes \big(\boldsymbol{\eta}^* \circledast \frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}}\big)\end{aligned}$$

where we have used eq. (B-17), and that

$$\boldsymbol{\eta} \equiv \boldsymbol{\eta}(\mathbf{q}) \quad \Rightarrow \quad \frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}} \equiv \frac{\partial\dot{\boldsymbol{\eta}}}{\partial\dot{\mathbf{q}}} \quad \text{and} \quad \frac{\partial\dot{\boldsymbol{\eta}}}{\partial\mathbf{q}} \equiv \frac{d}{dt}\big(\frac{\partial\dot{\boldsymbol{\eta}}}{\partial\dot{\mathbf{q}}}\big)$$

Therefore, using eq. (B-18), we can see that

$$\begin{aligned}\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\mathbf{q}} &= 2\boldsymbol{\eta}^* \circledast \frac{d}{dt}\big(\frac{1}{2}\boldsymbol{\eta} \circledast \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}}\big) - 2\big(\dot{\boldsymbol{\eta}} \otimes \boldsymbol{\eta}^*\big) \otimes \big[\boldsymbol{\eta}^* \circledast \big(\frac{1}{2}\boldsymbol{\eta} \circledast \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}}\big)\big] \\ &= \boldsymbol{\eta}^* \circledast \big[\dot{\boldsymbol{\eta}} \circledast \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}} + \boldsymbol{\eta} \circledast \frac{d}{dt}\big(\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}}\big)\big] - \frac{1}{2}\tilde{\boldsymbol{\omega}} \otimes \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}} \\ &= \frac{1}{2}\tilde{\boldsymbol{\omega}} \circledast \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}} + \frac{d}{dt}\big(\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}}\big) - \frac{1}{2}\tilde{\boldsymbol{\omega}} \otimes \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}} \\ \Rightarrow \quad \frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\mathbf{q}} &= \frac{d}{dt}\big(\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}}\big) + \frac{1}{2}\big([\tilde{\boldsymbol{\omega}}\circledast] - [\tilde{\boldsymbol{\omega}}\otimes]\big)\frac{\partial\tilde{\boldsymbol{\omega}}}{\partial\dot{\mathbf{q}}} \end{aligned} \tag{A-3}$$

From the last equation, we can immidiately see that

$$\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{q}} = \dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega \tag{A-4}$$

in which $\mathbf{J}_\omega$ is defined by

$$\mathbf{J}_\omega \triangleq \frac{\partial\boldsymbol{\omega}}{\partial\dot{\mathbf{q}}} \tag{A-5}$$

and $[\boldsymbol{\omega}\times]$ represents the cross-product matrix of $\boldsymbol{\omega}$. □

Similarly, one can show that a similar relation holds in the inertial frame.

Corollary 1. *The partial derivative of the angular velocity* ${}^{\mathcal{R}}\boldsymbol{\omega}$*, where* $\mathcal{R}$ *represents the inertial frame, of a rigid body in a serial kinematic chain with respect to the chain generalized-coordinate vector* $\mathbf{q}$ *can be expressed in terms of the Jacobian* ${}^{\mathcal{R}}\mathbf{J}_\omega$*, its time-rate, and the cross-product matrix* $[{}^{\mathcal{R}}\boldsymbol{\omega}\times]$ *of the angular velocity as*

$$\frac{\partial\,{}^{\mathcal{R}}\boldsymbol{\omega}}{\partial\mathbf{q}} = {}^{\mathcal{R}}\dot{\mathbf{J}}_\omega - [{}^{\mathcal{R}}\boldsymbol{\omega}\times]\,{}^{\mathcal{R}}\mathbf{J}_\omega. \tag{A-6}$$

Theorem 2. *The variation of the angular velocity* $\boldsymbol{\omega}$ *of a rigid body in a kinematic chain due to a virtual change in the chain generalized coordinates* $\mathbf{q}$ *can be expressed in terms of the Jacobian* $\mathbf{J}_\omega$*, its time rate, and the cross-product matrix* $[\boldsymbol{\omega}\times]$ *of the angular velocity as*

$$\big(\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{q}}\big)\delta\mathbf{q} = \big(\dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega\big)\delta\mathbf{q} \tag{A-7}$$

Proof: This theorem can essentially be proven the same way as Theorem 1. The only difference is that, because the generalized coordinates can be dependent, eq. (B-17) no longer holds. However, we can still relate the virtual changes $\delta\boldsymbol{\eta}^*$ and $\delta\boldsymbol{\eta}$ through

$$\frac{\partial\boldsymbol{\eta}^*}{\partial\mathbf{q}}\delta\mathbf{q} = -\boldsymbol{\eta}^* \otimes \big(\boldsymbol{\eta}^* \circledast \frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}}\big)\delta\mathbf{q} \tag{A-8}$$

which is basically the same equation projected along $\delta\mathbf{q}$.

Hence, we will have

$$\big(\frac{\partial\boldsymbol{\omega}}{\partial\mathbf{q}}\big)\delta\mathbf{q} = \big(\dot{\mathbf{J}}_\omega + [\boldsymbol{\omega}\times]\,\mathbf{J}_\omega\big)\delta\mathbf{q}$$

□

APPENDIX B

Let us consider two rotations represented by quaternions $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ and, at the same time, by rotation matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, respectively.

1. If these two rotations are performed one after the other in such a way that the resultant rotation $\mathbf{Q}_3$ is given by $\mathbf{Q}_3 = \mathbf{Q}_1\mathbf{Q}_2$, the compound quaternion $\boldsymbol{\eta}_3$ is given by

$$\boldsymbol{\eta}_3 = \boldsymbol{\eta}_1 \circledast \boldsymbol{\eta}_2 \equiv \boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_1 \tag{B-9}$$

where quaternion multiplication operations $\circledast$ and $\otimes$ are defined as below [19]:

$$[\boldsymbol{\eta}\circledast] \triangleq \begin{bmatrix} \eta_0\mathbf{1} + [\boldsymbol{\eta}_v\times] & \boldsymbol{\eta}_v \\ -\boldsymbol{\eta}_v^T & \eta_0 \end{bmatrix}, \qquad \text{and} \qquad [\boldsymbol{\eta}\otimes] \triangleq \begin{bmatrix} \eta_0\mathbf{1} - [\boldsymbol{\eta}_v\times] & \boldsymbol{\eta}_v \\ -\boldsymbol{\eta}_v^T & \eta_0 \end{bmatrix} \tag{B-10}$$

in which $\boldsymbol{\eta}_v$ and η_0 are the vector and scalar parts of the quaternion, respectively.

2. The two quaternion composition operators have associative properties, i.e.,

$$(\boldsymbol{\eta}_1 \circledast \boldsymbol{\eta}_2) \circledast \boldsymbol{\eta}_3 \equiv \boldsymbol{\eta}_1 \circledast (\boldsymbol{\eta}_2 \circledast \boldsymbol{\eta}_3) \tag{B-11}$$

$$(\boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_2) \otimes \boldsymbol{\eta}_3 \equiv \boldsymbol{\eta}_1 \otimes (\boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_3) \tag{B-12}$$

These properties can readily be verified in a symbolic manipulation software such as Maple.

3. Any quaternion has a unique conjugate $\boldsymbol{\eta}^* \equiv \begin{bmatrix} -\boldsymbol{\eta}_v^T & \eta_0 \end{bmatrix}^T$, so that

$$\boldsymbol{\eta} \circledast \boldsymbol{\eta}^* \equiv \boldsymbol{\eta}^* \circledast \boldsymbol{\eta} \equiv \boldsymbol{\eta} \otimes \boldsymbol{\eta}^* \equiv \boldsymbol{\eta}^* \otimes \boldsymbol{\eta} \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{B-13}$$

The partial derivatives of $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^*$ with respect to the set of generalized coordinates are related through

$$\frac{\partial\boldsymbol{\eta}^*}{\partial\mathbf{q}} = -\mathbf{D}\frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}} \tag{B-14}$$

where matrix $\mathbf{D}$ is defined as

$$\mathbf{D} \triangleq \begin{bmatrix} \mathbf{1}_3 & \mathbf{0} \\ \mathbf{0}^T & -1 \end{bmatrix} \tag{B-15}$$

It can readily be shown that $\mathbf{D}$ has the following properties

$$\mathbf{D}\,\boldsymbol{\eta}\circledast \equiv \boldsymbol{\eta}^* \otimes \mathbf{D} \quad \text{and} \quad \mathbf{D}\,\boldsymbol{\eta}\otimes \equiv \boldsymbol{\eta}^* \circledast \mathbf{D} \tag{B-16}$$

If the generalized coordinates are independent, then we will also have

$$\frac{\partial\boldsymbol{\eta}^*}{\partial\mathbf{q}} = -\boldsymbol{\eta}^* \otimes \big(\boldsymbol{\eta}^* \circledast \frac{\partial\boldsymbol{\eta}}{\partial\mathbf{q}}\big) \tag{B-17}$$

4. The time-derivative of the quaternion in the body-attached frame is given by [20]

$$\dot{\boldsymbol{\eta}} = \frac{1}{2}\,\tilde{\boldsymbol{\omega}} \otimes \boldsymbol{\eta} \equiv \frac{1}{2}\,\boldsymbol{\eta} \circledast \tilde{\boldsymbol{\omega}} \tag{B-18}$$

where the augmented angular velocity of the body is defined as $\tilde{\boldsymbol{\omega}} \triangleq \begin{bmatrix} \boldsymbol{\omega}^T & 0 \end{bmatrix}^T$. Using the properties 1, 2, and 4 above, we can solve eq. (B-18) for $\tilde{\boldsymbol{\omega}}$ as

$$\tilde{\boldsymbol{\omega}} = 2\dot{\boldsymbol{\eta}} \otimes \boldsymbol{\eta}^* \equiv 2\boldsymbol{\eta}^* \circledast \dot{\boldsymbol{\eta}} \tag{B-19}$$

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