Workspace Envelope Formulation of Planar Wire-Actuated Parallel Manipulators

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Abstract
In this article, the workspace of planar wire-actuated parallel manipulators is studied. The investigation is based on two methods: an analytical method which formulates the workspace envelope by means of Cramer's rule pertaining to the Jacobian matrix and the null space method from static analysis of manipulator. The workspace envelope method is also extended to analyzing redundant planar manipulators and planar manipulators with an external wrench or gravity modelled as an additional wire. It is discussed that the null space method gives a more realistic workspace formulation as it takes into account wire tension limits, while the workspace generated by the workspace envelope method assumes very large wire tensions are possible. The workspace envelope method plots an analytical function as the border of the workspace so a much higher resolution representation of the wrench closure workspace is possible.

Keywords: wire-actuated parallel manipulators, wrench closure workspace, workspace envelope

1 INTRODUCTION

The workspace analysis is a key requirement in the design of robot manipulators as it identifies the set of locations in which a robotic manipulator can operate. Without defining the workspace, the operator would not be able to properly use the manipulator, constantly attempting to position the manipulator in poses (positions and orientations) not actually reachable by the manipulator while maintaining control. The workspace of planar wire-actuated parallel manipulators is formulated in this paper. Parallel manipulators consist of closed loop of links connected together with joints. The legs or branches of a parallel manipulator are kinematic chains of links and joints that connect the base platform to the mobile platform (end effector). A manipulator is called planar if all the rigid bodies are confined to motion on a plane or parallel planes.

If each leg of a planar parallel manipulator is replaced by a single wire, the manipulator is referred to as a planar wire-actuated (or wire-suspended) parallel manipulator. Examples of these manipulators can be seen in Figure 1. Actuated spools at the base are the most common method of controlling the wires. The pose of the mobile platform is controlled by manipulating the lengths of the wires. There are many advantages of using wires over rigid body linkages, such as the light weight and long range of wires which allow high speed motion and large workspaces.

Similar to constraining an object to planar motion with three degrees of freedom (DOF), two translations and one rotation, with four frictionless contact points [1], at least four wires are required to control a planar manipulator with the same three degrees of freedom.

The set of locations where a wire-actuated manipulator has force/moment (wrench) closure, i.e., when all wire tensions are positive for any given external wrench applied to the mobile platform, is the wrench closure workspace (WCW). Discrete workspace models based on wire tension have been investigated for wire-actuated parallel manipulators, e.g., [2]-[4]. Analytical methods, which define the workspace boundary, have also been presented, e.g., [5]-[6]. The discrete workspace models based on wire tension can account for external wrenches acting on the mobile platform, such as gravity, while the analytical models presented do not. The minimum and maximum tensions of the wires are accounted for only in these discrete workspace models. An iterative calculation of wire tensions method (based on the null space of Jacobian matrix) and an algebraic (convex hull) method of calculating the workspace of planar wire-actuated parallel manipulators was investigated in [7]. Methods based on the antipodal theorem from grasping manipulators were presented and applied to the non-redundant planar wire-actuated parallel manipulators in the absence and presence of gravity and external wrench in [8]-[9].

The workspace of planar wire-actuated parallel manipulators, based on the wire tension and also using an analytical formulation of the null space of the transposed Jacobian will be investigated in this article. The manipulator parameters and workspace formulation based on wire tensions are reported in Section 2. A technique based on Cramer's rule for formulating the workspace is presented in Section 3. The simulation for both methods is discussed in Section 4. The conclusions of the article are stated in Section 5.

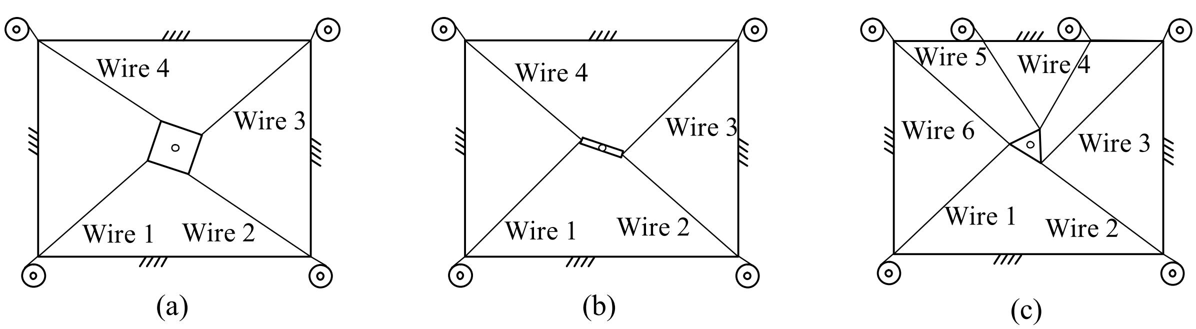


Figure 1 Example planar wire-actuated parallel manipulators.

2 KINEMATIC AND STATIC MODELS

To investigate the workspace of a manipulator, fixed and moving coordinate systems are assigned. The coordinates and parameters for a planar wire-actuated parallel manipulator are shown in Figure 2. The base has the fixed coordinate system $\Psi(X, Y)$ located at 0, while the mobile platform has a coordinate system, $\Gamma(X', Y')$, fixed to the center of mass point P; P has coordinates (x, y) in $\Psi(X, Y)$. The mobile platform is connected to the base by n wires each with length l_i. The attachment points of the wires to the base platform, anchors, are denoted A_i, while the attachment points on the mobile platform are denoted B_i. The angle at which the mobile platform is oriented with respect to $\Psi(X, Y)$ is given as φ and the orientation of lines $\overline{PB}_i$ with respect to the mobile platform frame $\Gamma(X', Y')$ are given by angles θ_i. For the following analysis, all of these parameters are assumed to be known.

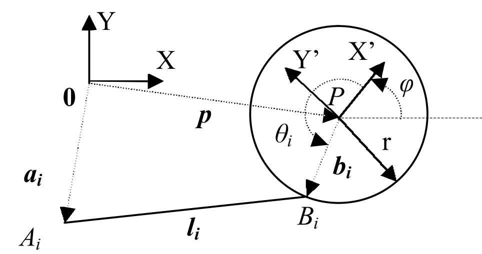


Figure 2 Coordinates and variables for planar wire-actuated parallel manipulators.

The position vector of point A_i with respect to the fixed frame is ${}^{\Psi}\mathbf{a}_i = [\,a_{i_x} \quad a_{i_x}\,]^T$, and the position vector of point B_i with respect to $\Gamma(X', Y')$ is given as ${}^{\Gamma}\mathbf{b}_i = [\,b_i \cos\theta_i \quad b_i \sin\theta_i\,]^T$, $i = 1, \ldots, n$, where b_i is the length of the line segment $\overline{PB_i}$. To find the vector of the magnitude and direction of each wire,

$$ {}^{\Psi}\mathbf{l}_i = {}^{\Psi}\mathbf{a}_i - {}^{\Psi}\mathbf{b}_i = \begin{bmatrix} l_{i_x} \\ l_{i_y} \end{bmatrix} = \begin{bmatrix} a_{i_x} - x - b_i \cos\theta_i \cos\varphi + b_i \sin\theta_i \sin\varphi \\ a_{i_y} - y - b_i \cos\theta_i \sin\varphi - b_i \sin\theta_i \cos\varphi \end{bmatrix}, \quad i = 1, \ldots, n \tag{1} $$

To determine the wire forces, τ_i, the static force balance is required as follows.

$$ \sum F_x = F_{ext\_x} \Rightarrow \sum_{1}^{n} \tau_i \cos\alpha = F_{ext\_x} \tag{2.1} $$

$$ \sum F_y = F_{ext\_y} - mg \Rightarrow \sum_{1}^{n} \tau_i \sin\alpha = F_{ext\_y} - mg \tag{2.2} $$

$$ \sum M_z = M_{ext\_z} \Rightarrow \sum_{1}^{n} \tau_i v_i = M_{ext\_z} \tag{2.3} $$

where $\cos\alpha_i = l_{ix} / \|\mathbf{l}_i\|$, $\sin\alpha_i = l_{iy} / \|\mathbf{l}_i\|$, m is the mass of the mobile platform, $g = 9.81$ m/s^2 is the gravitational constant, F_{ext_x}, F_{ext_y} and M_{ext_z} are the external force and moment acting on the mobile platform, and $\boldsymbol{v}_i$ is the normal distance from point P to each wire axis. Using matrix notation, force and moment balance equations (2.1)-(2.3) can be written in terms of the transpose of Jacobian matrix

$$ \mathbf{F} = \mathbf{J}^T \boldsymbol{\tau} \tag{3} $$

where $\mathbf{F} = \left[F_{ext\_x} \quad \left(F_{ext\_y} - mg\right) \quad M_{ext\_z}\right]^T$ is the static wrench of mobile platform, $\boldsymbol{\tau} = \left[\tau_1 \quad \cdots \quad \tau_n\right]^T$ is the vector of wire tensions, and the 3xn $\mathbf{J}^T$ matrix, n>3, is

$$\mathbf{J}^T = \begin{bmatrix} \cos\alpha_1 & \cdots & \cos\alpha_n \\ \sin\alpha_1 & \cdots & \sin\alpha_n \\ \left({}^{\Psi}b_{1_x} - x\right)\sin\alpha_1 - \left({}^{\Psi}b_{1_y} - y\right)\cos\alpha_1 & \cdots & \left({}^{\Psi}b_{n_x} - x\right)\sin\alpha_n - \left({}^{\Psi}b_{n_y} - y\right)\cos\alpha_n \end{bmatrix} \tag{4}$$

For a given $\mathbf{F}$, the wire tensions need to be positive for the mobile platform to be in the feasible workspace. Because $\mathbf{J}^T$ is not square its generalized inverse (Moore-Penrose inverse), $\mathbf{J}^{\#T} = \mathbf{J}(\mathbf{J}^T\mathbf{J})^{-1}$, is used to solve for force analysis.

$$\boldsymbol{\tau} = \mathbf{J}^{\#T}\mathbf{F} + \left(\mathbf{I} - \mathbf{J}^{\#T}\mathbf{J}^T\right)\boldsymbol{\lambda} \tag{5}$$

where $\mathbf{J}^{\#T}\mathbf{F}$ is the minimum norm or particular solution and the second term is the homogenous solution in which $(\mathbf{I}-\mathbf{J}^{\#T}\mathbf{J}^T)$ maps the free vector $\boldsymbol{\lambda}$ to the null space of $\mathbf{J}^T$.

The free vector, $\boldsymbol{\lambda}$, is chosen such that positive tension is maintained in wires. Testing each location and choosing a $\boldsymbol{\lambda}$ that results in $\tau_i>0$, $i = 1, \ldots, n$ guarantees that the location is inside the workspace. This is referred to as the null space method for calculating the workspace.

The calculation of $\boldsymbol{\lambda}$ depends on whether or not the manipulator has redundancy and/or is under the action of external wrench. For the case without redundancy, i.e., when the null space basis is defined by a vector, and there is no external wrench, $\mathbf{J}^{\#T}\mathbf{F} = \mathbf{0}$, then in equation (5) the wire tensions are calculated based only on the null space. Thus, all the entries of the vector that spans the null space should be either positive or negative so that a scalar λ can be chosen that results in all positive tensions. For the case without redundancy and a non-zero external wrench, the null space vector may have positive and negative entries. The $\boldsymbol{\lambda}$ is chosen by comparing the positive and negative values of the minimum norm solution to the entries in the null space vector to determine if it is possible for all wire tensions to be positive. For the case that the manipulator has redundancy, the null space is spanned by multiple vectors. If an external wrench is applied to the redundant manipulator, it is possible to identify the entries of $\boldsymbol{\lambda}$ by finding the intersections of the functions described by the rows of the matrix equation (5) because the minimum norm solution is non-zero. For the case of redundancy, in the absence of an external wrench, without loss of generality one entry of $\boldsymbol{\lambda}$ is assumed to be one and the other entries of $\boldsymbol{\lambda}$ can be solved by finding the intersections of the functions described by the rows of the matrix equation (5).

3 WORKSPACE ENVELOPE CHARACTERIZATION

The workspace boundary of a manipulator is defined as a set of singularities that enclose the area which has feasible wire tensions. Thus, it is possible to determine the workspace of a manipulator without examining all potential locations. The determinants of the minors of the Jacobian matrix to define the workspace envelope of wire-actuated parallel manipulators were investigated in [5] and [6].

The transposed Jacobian of a wire-actuated parallel manipulator with m degrees of freedom and n wires will have a null space spanned by $n-m$ vectors with dimension $n \times 1$. When $n = m+1$ the null space is spanned by a single $n \times 1$ vector and if all the entries of that null space vector are strictly positive or all the entries are strictly negative then the wire tensions can be positive for any external wrench. This is because, for any value of the particular solution, there exists a value of $\boldsymbol{\lambda}$ that can generate positive tensions, including poses that require very large wire tension. The collection of manipulator poses that generate all positive or all negative null space entries comprise the WCW because any external wrench can be withstood by the manipulator. For a specific external wrench applied to the mobile platform, there may exist a null space vector that has both positive and negative values which can generate positive wire tensions. Hence, for this case the workspace formulated by identifying the loci that correspond to null space vectors with all positive or all negative entries may not be the complete workspace of the manipulator.

When $n = m+1$ the manipulator will have the minimum number of wires required and will be referred to as "non-redundant". In this case, Cramer's rule, [10], for the solution of linear matrix equations could be applied to calculate the $n \times 1$ null space vector of the transposed Jacobian in terms of the pose of the mobile platform (location x and y, and orientation φ). For a planar wire-actuated parallel manipulator with $n = m+1$ wires, in the absence of external wrench, using $\mathbf{C}_i$ as the i^{th} column of the transposed Jacobian, where $\mathbf{C}_i$ is a 3×1 vector , equation (3) reduces to

$$\begin{bmatrix} \mathbf{C}_1 & \cdots & \mathbf{C}_n \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = \tau_1 \mathbf{C}_1 + \ldots + \tau_j \mathbf{C}_j + \ldots + \tau_n \mathbf{C}_n = \mathbf{0}, \tag{6}$$

The j^{th} wire tension, τ_j, could be set to one and the expression be rearranged so a square matrix is formed, e.g., by setting τ_n to one

$$\begin{bmatrix} \mathbf{C}_1 & \cdots & \mathbf{C}_{n-1} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_{n-1} \end{bmatrix} = -\mathbf{C}_n, \tag{7}$$

By replacing τ_j, Cramer's rule can be used to solve for tensions τ_i, for $i = 1,\ldots,n$,

$$\tau_i = \frac{\left| [\mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{j-1} \; \mathbf{C}_{j+1} \cdots \mathbf{C}_{i-1} \; -\mathbf{C}_j \; \mathbf{C}_{i+1} \cdots \mathbf{C}_n] \right|}{\left| [\mathbf{C}_1 \; \cdots \; \mathbf{C}_{j-1} \; \mathbf{C}_{j+1} \; \cdots \; \mathbf{C}_n] \right|}, \; i=1,\ldots,n \text{ and } j \neq i \tag{8}$$

It should be noted that τ_j can be written as

$$\tau_j = 1 = \frac{\left| [\mathbf{C}_1 \; \cdots \; \mathbf{C}_{j-1} \; \mathbf{C}_{j+1} \; \cdots \; \mathbf{C}_n] \right|}{\left| [\mathbf{C}_1 \; \cdots \; \mathbf{C}_{j-1} \; \mathbf{C}_{j+1} \; \cdots \; \mathbf{C}_n] \right|} \tag{9}$$

Changing the sign of a column or switching the location of any two columns, the sign of the determinant is inverted resulting in the following expression for τ_i, when $\tau_j = \tau_n$.

$$\tau_i = \frac{(-1)^{n-i} \left| [\mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{i-1} \; \mathbf{C}_{i+1} \cdots \; \mathbf{C}_n] \right|}{\left| [\mathbf{C}_1 \; \cdots \; \mathbf{C}_{n-1}] \right|}, \; i=1,\ldots,n-1 \tag{10}$$

Then, the submatrices (minors) of $\mathbf{J}^T$ can be denoted $\mathbf{M}_i = [\mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{i-1} \; \mathbf{C}_{i+1} \cdots \mathbf{C}_n]$. Since the denominator is the same for all τ_i it can be factored out from the entries of vector $\boldsymbol{\tau}$. Cancelling out this factor would result in a vector, denoted as $\mathbf{K}$, that spans the null space of the transposed Jacobian because $\mathbf{J}^T \boldsymbol{\tau} = \mathbf{0}$.

$$\mathbf{K} = \left[(-1)^{n-1} \det(\mathbf{M}_1) \quad (-1)^{n-2} \det(\mathbf{M}_2) \quad \cdots \quad (-1)\det(\mathbf{M}_m) \quad \det(\mathbf{M}_n) \right]^T \tag{11}$$

By setting each entry of $\mathbf{K}$ in equation (11) as an inequality, either greater than or less than zero, it is possible to plot the loci of poses which generate a positive or negative value for that entry of $\mathbf{K}$ that spans the null space. The inequalities are set up as

$$\mathbf{K} = \begin{bmatrix} (-1)^{n-1} \det(\mathbf{M}_1) > 0 \\ (-1)^{n-2} \det(\mathbf{M}_2) > 0 \\ \vdots \\ \det(\mathbf{M}_n) > 0 \end{bmatrix} \text{or } \mathbf{K} = \begin{bmatrix} (-1)^{n-1} \det(\mathbf{M}_1) < 0 \\ (-1)^{n-2} \det(\mathbf{M}_2) < 0 \\ \vdots \\ \det(\mathbf{M}_n) < 0 \end{bmatrix} \forall (x, y, \varphi) \in \text{WCW} \tag{12}$$

which describe the conditions on the entries of the null space for poses that are contained within the WCW for an n-wire manipulator with $n = m+1$ wires.

The algorithm which calculates the WCW starts by formulating the transposed Jacobian in terms of the mobile platform pose. The planar manipulator is then checked for redundancy, if the manipulator is not redundant ($n = m+1$) then the determinants of the $m \times m$ minors of the transposed Jacobian are calculated for equation (12). Each inequality is plotted to find the region for which that entry of the null space vector is strictly positive. Once this is done for all entries of the null space vector, the intersection between the positive inequalities is retained. This intersection region generates strictly positive values for all entries in the null space vector. This is repeated for the case that all the entries of the null space vector are strictly negative. The union of the strictly positive intersection region and the strictly negative intersection region is the WCW of the non-redundant manipulator. The curves which define the border of the workspace, functions obtained by changing the inequalities in equation (12) to equalities, are quadratic functions in terms of variables x and y, the position of the mobile platform. The constant orientation WCW is formulated by keeping a constant mobile platform orientation, φ.

When $n > m+1$, the manipulator will be referred to as "redundant". For this case, the degree of redundancy is calculated and the $m \times m+1$ submatrices of the $m \times n$ transposed Jacobian are formulated. The process of identifying the contribution to the WCW of each $m \times m+1$ submatrix is the same as the process for non-redundant planar manipulators. Once the contribution of each $m \times m+1$ submatrix is formulated and retained, the union of the contributions of all the $m \times m+1$ submatrices is the complete WCW of a planar manipulator. For the planar wire-actuated parallel manipulators investigated in this paper, the simulation indicated that considering the 3×4 submatrices of the transposed Jacobian correctly identifies the WCW. For general wire-actuated parallel manipulators, investigating only the $m \times m+1$ submatrices may not result in the entire workspace [5].

This method of formulating the WCW of planar wire-actuated parallel manipulators by investigating the analytical solution for the null space of the transposed Jacobian will be referred to as workspace envelope characterization. In the following section, the workspace envelope characterization and the null space method of Section 2 will be used to investigate the workspace of the manipulators in Figure 1.

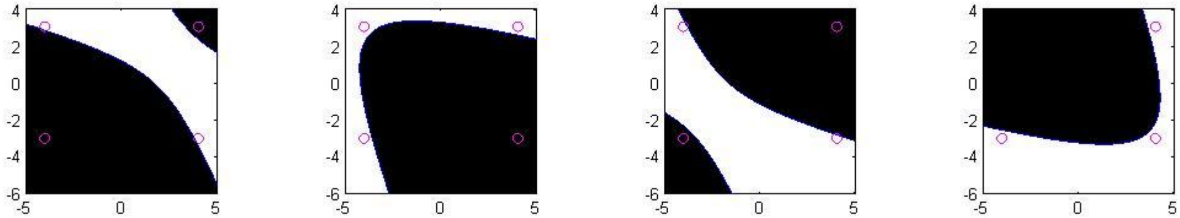
4 SIMULATION

In this section, the workspace of the planar wire-actuated parallel manipulators of Figure 1 is investigated using the workspace envelope characterization and the null space method. In the following simulations, the plane of motion is the horizontal plane, i.e., gravity does not affect the motion of the manipulator, unless otherwise stated.

4.1 Non-Redundant Manipulators

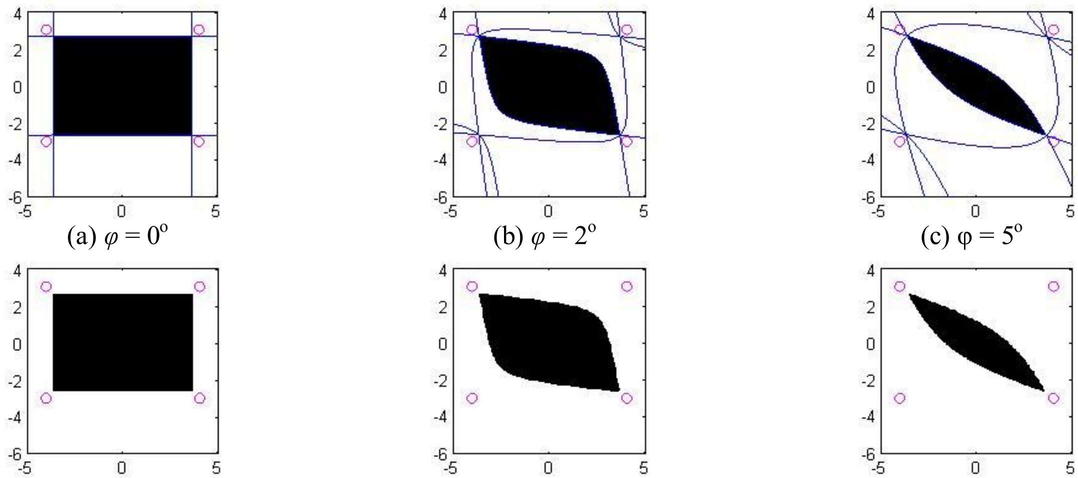
The WCW of non-redundant ($n = m+1$) manipulators is formulated in this section. Figure 3 shows the plots of inequalities of equation (12) for the manipulator in Figure 1(a), $n = 4$ and $m = 3$, with a mobile platform orientation of $\varphi = 5^{\circ}$ and no external wrench acting on the mobile platform. The case shown in Figure 3 has all equations strictly greater than zero. For the case that all entries of the null space are strictly less than zero, there is an empty intersection between inequalities, thus the constant orientation WCW is formulated from only those plots shown in Figure 3. In the plots of Figure 3 and following figures, the anchor positions are represented by small circles. The complete workspace, i.e., the intersection of regions that generate positive entries of the null space (plots of Figure 3), can be seen in Figure 4(c).

Figures 4(a) and 4(b) depict the constant orientation WCW of the manipulator in Figure 1(a) calculated by finding the borders of the workspace with no external wrench acting on the mobile platform at constant orientations of $\varphi = 0^{\circ}$ and 2° respectively. The lines in each plot of Figure 4 show the limits of inequalities in equation (12). The black shaded region is the workspace of the manipulator at the given orientation. Figures 4(d)-4(f) show the workspace plots of this manipulator formulated using the null space method with no external wrench at orientations of $\varphi = 0^{\circ}$, 2°, 5° respectively. For the null space method, the minimum and maximum allowable wire tensions are set to zero and 525N respectively. If a lower limit is considered for the maximum allowable wire tension then the workspace obtained using the null space method would be smaller.



(a) $(-1)^3 \det(\mathbf{M}_1) > 0$ (b) $(-1)^2 \det(\mathbf{M}_2) > 0$ (c) $(-1)\det(\mathbf{M}_3) > 0$ (d) $\det(\mathbf{M}_4) > 0$

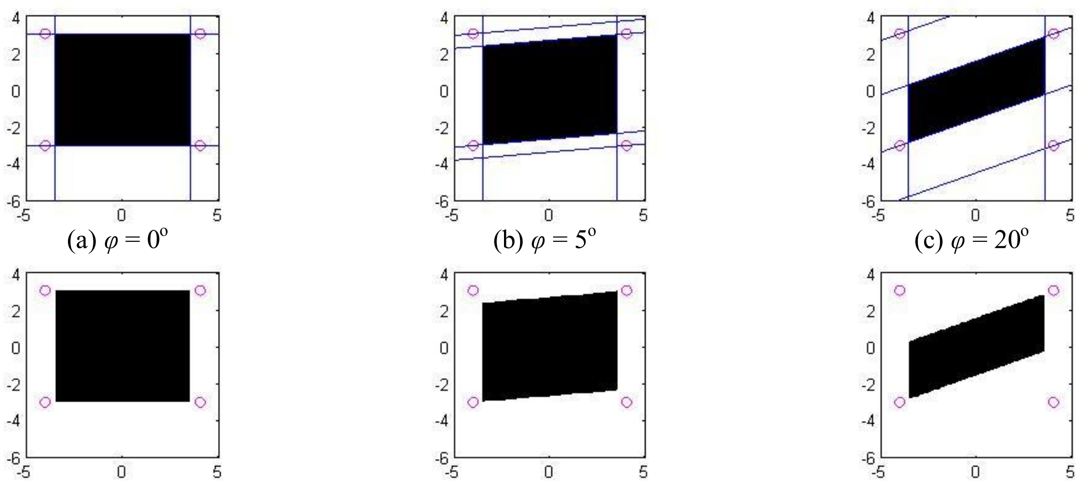
Figure 3 Regions enclosed by positive inequalities of equation (12) for manipulator in Figure 1(a) at $\varphi = 5^{\mathrm{o}}$.



(a) $\varphi = 0^{\mathrm{o}}$ (b) $\varphi = 2^{\mathrm{o}}$ (c) $\varphi = 5^{\mathrm{o}}$

(d) $\varphi = 0^{\mathrm{o}}$ (e) $\varphi = 2^{\mathrm{o}}$ (f) $\varphi = 5^{\mathrm{o}}$

Figure 4 Workspace of manipulator in Figure 1(a): (a)-(c) envelope characterization method, (d)-(f) null space method.



(a) $\varphi = 0^{\mathrm{o}}$ (b) $\varphi = 5^{\mathrm{o}}$ (c) $\varphi = 20^{\mathrm{o}}$

(d) $\varphi = 0^{\mathrm{o}}$ (e) $\varphi = 5^{\mathrm{o}}$ (f) $\varphi = 20^{\mathrm{o}}$

Figure 5 Workspace of manipulator in Figure 1(b): (a)-(c) envelope characterization method, (d)-(f) null space method.

Figures 5(a)-5(c) depict the constant orientation WCW plots of the manipulator in Figure 1(b) using equation (12) with no external wrench acting on the mobile platform at orientations of $\varphi = 0^o$, 5^o, 20^o respectively. Figures 5(d)-5(f) show the workspace plots of this manipulator formulated using the null space method with no external wrench at orientations of $\varphi = 0^o$, 5^o, 20^o respectively. Any difference between the plots obtained with the workspace envelope characterization and the plots obtained with the null space method is from the minimum and maximum wire tensions that are taken into account in the null space method and also the high resolution possible with the envelope characterization method.

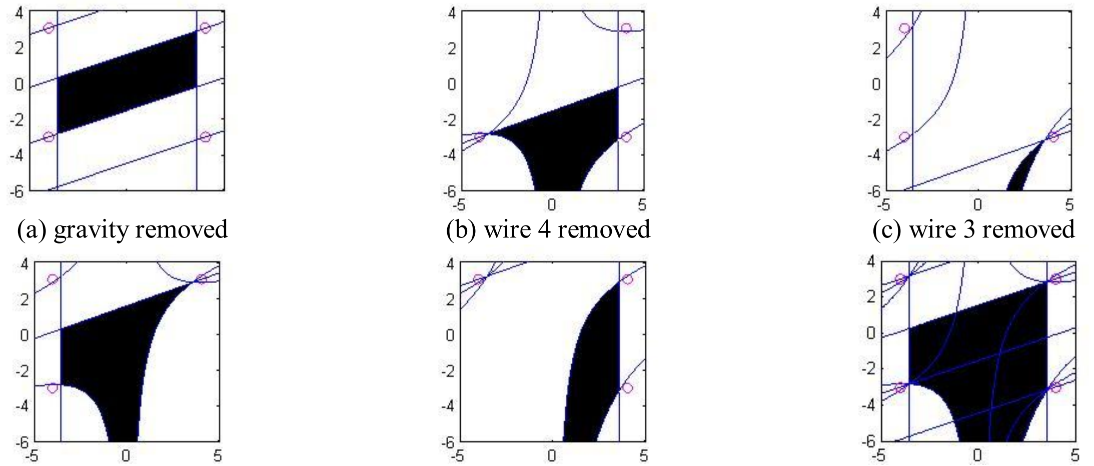
4.2 Manipulator with External Wrench

The effect of gravity and external wrench on the workspace of the manipulator in Figure 1(b) is investigated in this section. Only the manipulator in Figure 1(b) is investigated in this section because the workspace of this manipulator is larger than the workspace of the manipulator in Figure 1(a) and larger orientations are possible for this manipulator compared to the manipulator in Figure 1(a). When using the workspace envelope characterization it is possible to model gravity as a wire acting on the mobile platform. By modelling gravity as a wire with constant force and constant direction, a four-wire-actuated parallel manipulator is modelled as a five-wire manipulator. The Jacobian is modified to include an extra column for a wire which represents the gravitational force and is constantly pulling in the negative Y-direction. The magnitude of the gravitational term is taken as the "tension" and the column added to the transposed Jacobian is $[0\ -1\ 0]^T$. Since the workspace envelope characterization only investigates properties of the Jacobian matrix it does not take into account the magnitude of the gravitational force. Therefore, the workspace results for a manipulator with gravity are the same as the results of the workspace with a large external force in the negative Y-direction using this workspace envelope formulation. The simulations using this method would result in a workspace larger than the actual workspace for the manipulators with low specific mass for mobile platform or manipulators operating in environments where the gravitational acceleration is not as high, so the weight (*mg*) of the mobile platform will be low

For a five-wire planar manipulator, the workspaces of the five different combinations of four wires are investigated. The union of the workspaces of the four-wire combinations is the workspace of the "five-wire" manipulator. The 3 x 4 submatrices of the transposed Jacobian correctly formulate the WCW of this manipulator. For this WCW, the four-wire manipulator can withstand any external wrench only if the specific wrench (modelled as a wire) is applied to the manipulator.

Figure 6 shows the WCW corresponding to four wires/forces at a time formulated using equation (12) for the four-wire manipulator in Figure 1(b) under the action of gravity. Each plot in Figure 6 depicts the anchor positions corresponding to the wires being investigated for that plot. For the plots of Figure 6, all intersections of the inequalities that are less than zero are empty, so the WCW is formulated from only the inequalities that are greater than zero. The orientation of the mobile platform is 20^o for the plots in Figure 6. The union of the workspaces in Figures 6(a)-6(e), shown in Figure 6(f), depict the WCW of the manipulator in Figure 1(b) with a large external force in the negative Y-direction. The WCW in Figure 6(f) has a shape that tapers to infinity in the negative Y-direction. As the mobile platform descends in the negative Y-direction, the wires become parallel at infinity and the manipulator will be uncontrollable. Figures 7(a)-7(c) depict the WCW for the manipulator in Figure 1(b) with gravity obtained at orientations of $\varphi = 0^o$, 5^o, and 30^o. Figures 7(d)-7(f) show the workspaces for this manipulator with gravity formulated with the null space method at orientations of $\varphi = 0^o$, 5^o, and 30^o. Any difference between the workspace plots of the two methods is due to the wire tension limits that the null space method takes into account and also the high resolution possible with the plots of envelope characterization method.

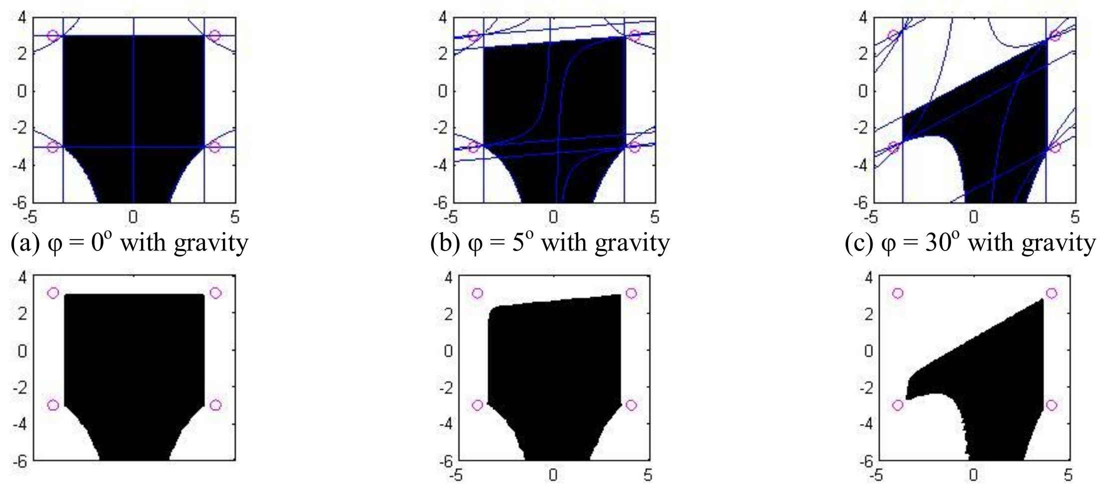
The manipulator in Figure 1(b) was also investigated with a large force acting in the positive X-direction and a large positive moment acting on the mobile platform by inserting columns $[1\ 0\ 0]^T$ or $[0\ 0\ 1]^T$ into the transposed Jacobian. Using any normalized direction vector for the first and second entries of the extra column allows the workspace to be calculated with a large external force acting in that direction, such as $[0.707\ 0.707\ 0]^T$ for an external force acting at 45° counter-clockwise from the X-axis. The WCW formulated with workspace envelope characterization method for these cases can be seen in Figures 8(a)-8(c), the workspace calculated with the null space method, with an external force of 100N or an external moment of 100Nm are shown in Figures 8(d)-8(f). The orientation of the mobile platform is $\varphi = 20°$ for all plots in Figure 8.



(a) gravity removed (b) wire 4 removed (c) wire 3 removed

(d) wire 2 removed (e) wire 1 removed (f) Complete workspace

Figure 6 Workspaces corresponding to combinations of four forces for the manipulator in Figure 1(b) acting under gravity at $\varphi = 20°$.



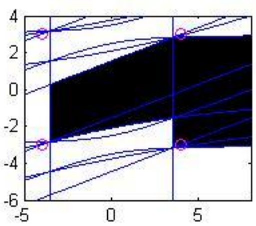
(a) $\varphi = 0°$ with gravity (b) $\varphi = 5°$ with gravity (c) $\varphi = 30°$ with gravity

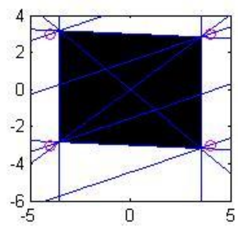
(a) $\varphi = 0°$ with gravity (b) $\varphi = 5°$ with gravity (c) $\varphi = 30°$ with gravity

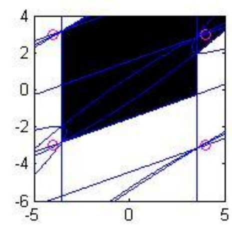
Figure 7 Workspaces of manipulator in Figure 1(b) with gravity: (a)-(c) envelope characterization method, (d)-(f) null space method.

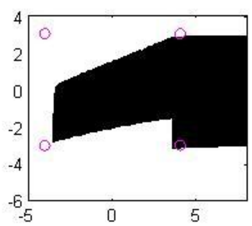
By applying an external wrench to the mobile platform, the workspace can be altered to have more desirable characteristics. Comparing the WCW plot of Figure 5(c), formulated with a zero external wrench, and the WCW of Figure 8(b), formulated with a large external moment, which are calculated for the same manipulator and mobile platform orientation of $\varphi = 20°$, there is a large difference in the workspace shape and size. The workspace of Figure 8(b) is much larger and thus desirable compared to the plot of Figure 5(c) , and as depicted, it is similar to the workspace of Figure 5(b) with orientation of φ

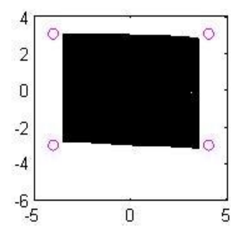
= 5°. By applying an external force in a certain direction, the workspace will increase in that direction, as can be seen in Figures 8(a) and 8(c). Applying an external moment in the same direction of mobile platform orientation increases the workspace until it is similar to the zero orientation workspace, as depicted in Figure 8(b), the larger the mobile platform orientation the greater the increase in workspace area. If the mobile platform orientation is zero there is no increase in workspace size with an external moment.


(a)Large force in X-direction, envelope characterization method


(b) Large positive moment, envelope characterization method


(c) Large force at 45° to X-axis, envelope characterization method


(d) 100N force in X-direction, null space method


(e) 100Nm positive moment, null space method

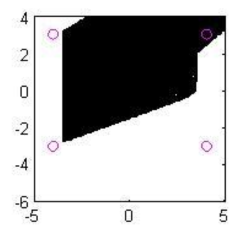

(f) 100N force acting at 45° to X-axis, null space method

Figure 8 Workspaces for large external force and moment acting on the mobile platform $\varphi = 20^o$.

The tapering of the WCW for the large force acting in the negative Y-direction, i.e., gravity, and lack of tapering for the WCW for the large force acting in the positive X-direction is affected by the shape of the mobile platform. Since the mobile platform is in a horizontal position at zero orientation and the wire connection points on the mobile platform are one unit apart along the X-axis, when the manipulator moves along the Y-axis beyond the lower anchor positions the workspace tapers because the wires approach a parallel configuration. When the mobile platform moves along the X-axis the wire connection points on the mobile platform are at the same point along the Y-axis, thus the mobile platform must be at a location much further along the X-axis for the wires to approach a parallel configuration.

4.3 Redundant Manipulator ($n>m+1$)

To further investigate the affect of redundancy on the workspace of wire-actuated parallel manipulators, equation (12) is used to formulate the workspace of the manipulator in Figure 1(c). Since this manipulator has six wires there are a total of 15 combinations of four wires that generate overlapping sections of the workspace of redundant manipulator.

Figure 9 depicts the workspaces of each of the 15 four-wire combinations for a mobile platform orientation of $\varphi = 20^o$ and with zero external wrench. The set of anchors that are used for each combination are shown in each plot. Each plot of Figure 9 is formed from the union of the regions that is the intersection of all positive entries of the null space vector, as with previous cases the intersection of all the negative entries of the null space vector is empty. The union of these 15 combinations can be seen in Figure 10(c) resulting in the WCW of the manipulator in Figure 1(c) at $\varphi = 20^o$. Figures 10(a) and 10(b) show the complete WCW of the manipulator in Figure 1(c) at orientations of $\varphi = 0^o$ and $\varphi = 5^o$. Figures 10(d)-(f) depict the workspace plots of this manipulator formulated with the null space method at orientations $\varphi = 0^o$, 5^o, and 20^o respectively. Again, any difference between the workspace plots of the two methods is due to the wire tension limits that the null space method takes into account and also the high resolution possible with the plots of envelop characterization method.

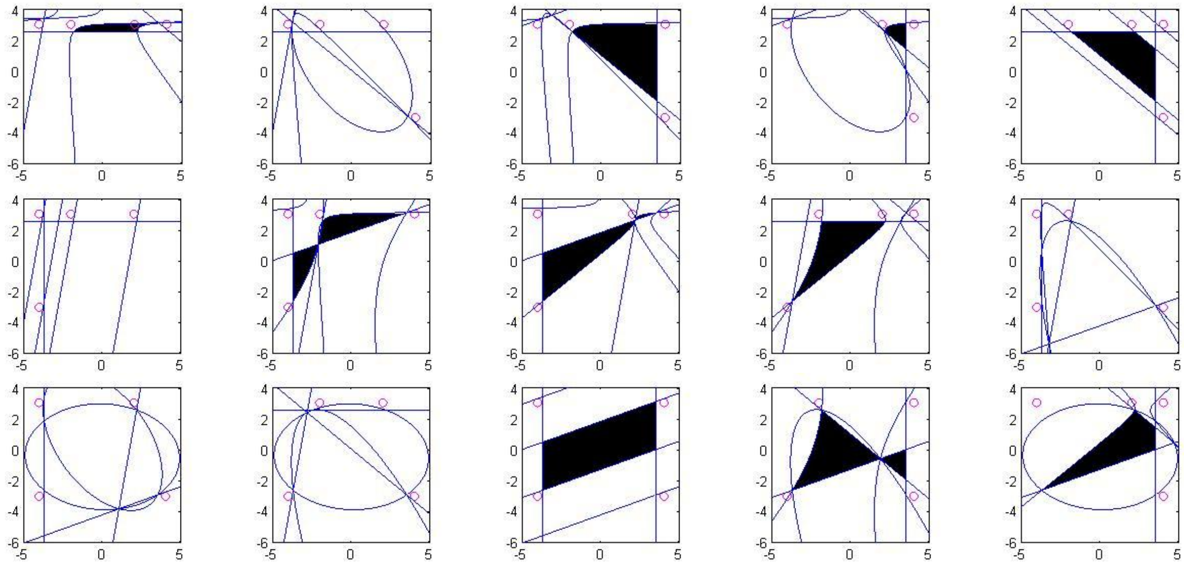


Figure 9 WCW using workspace envelope characterization of four wires of manipulator in Figure 1(c) at $\varphi = 20^o$.

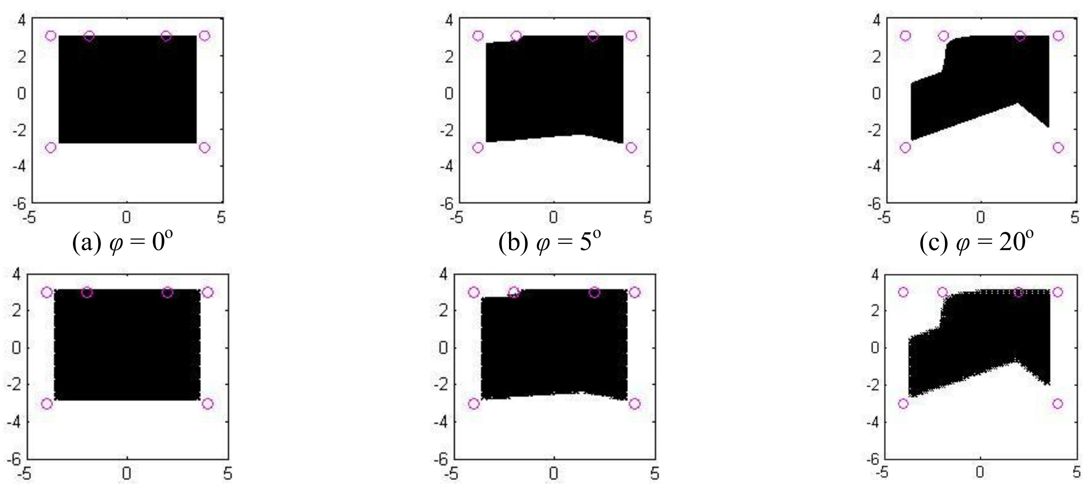


Figure 10 Workspaces of manipulator in Figure 1(c).

5 CONCLUSION

In this article, it was presented that the workspace of 3 DOF planar wire-actuated parallel manipulators can be formulated utilizing an analytical method which describes the border of the workspace. The proposed method is based on the determinants of the 3 x 3 minors of the transposed Jacobian that characterize the null space of the Jacobian matrix. The intersection of the regions that generate all positive determinants of the 3 x 3 minors and intersection of the regions that generate all negative determinants form the workspace of a planar manipulator with four wires. It should be noted that for all cases tested in this paper, the intersection of regions that generate all negative determinants were empty, thus all workspaces were formulated from the intersection of the regions that generate positive determinants. The wrench closure workspace, with constant orientation of the mobile platform, was calculated for three manipulators: two non-redundant four-wire-actuated manipulators and one six-wire-actuated manipulator. For redundant manipulators, the workspace was investigated by extending the workspace envelope characterization method and investigating the contribution of each set of four wires. Through simulation, the complete WCW of the manipulators presented in this paper were obtained by considering the 3 x 4 submatrices of the transposed Jacobian. In general, investigating the $m \times m+1$ submatrices may not be sufficient to generate the complete WCW. It was shown that modelling gravity/large external wrench as a wire and investigating each combination of four "wires" allowed the WCW formulation of manipulators acting under gravity/large external wrenches. In these WCW, the four-wire manipulator can withstand any external wrench only if the specific wrench (modelled as a wire) is applied to the manipulator.

Applying a large external wrench on the mobile platform drastically changes the shape and size of the workspace. The plots of the WCW with a large external force will be larger in the direction of that force. By applying a large external moment on the mobile platform, the workspace region does not change for a zero mobile platform orientation. The greater the orientation of the mobile platform is, the greater the increase in the size of the workspace will be when a moment in the same direction as mobile platform orientation is applied. It should be noted that a moment in opposite direction of orientation will result in a smaller workspace. By applying an external wrench to the mobile platform, it is possible to increase the size and/or change the shape of the workspace accordingly.

All workspaces calculated with the workspace envelope method match workspaces formulated with the null space method. The null space method gives a more realistic workspace formulation because it takes into account the minimum and maximum wire tensions. The workspace envelope method assumes that very large wire tensions is possible and identifies the maximum wrench closure workspace. Since the workspace envelope method plots an analytical function as the border of the workspace a much higher resolution representation of the workspace is possible compared to the null space method. There is no significant difference between the calculation speeds of the two methods. The null space method is more useful when investigating the design constraints of a manipulator to be manufactured while the workspace envelope method identifies the maximum possible workspace where any external wrench can be resisted.

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