Topological Synthesis of Translational Parallel Manipulators

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Abstract

For translational parallel manipulators (TPM), topology synthesis methods that can be found in the literature are mainly based on screw theory, instantaneous kinematics, or group theory. In this work, finite displacement equations are used for the topology synthesis of TPM. Serial chains with less than 6 degrees of freedom (DOF) are first investigated and, topological conditions for them to generate 3D translations while its end-effector (EE) is under a constant orientation constraint are derived. Then the parallel manipulators (PM) composed of these serial chains are analyzed to find out whether and under what conditions the EE will keep a constant orientation throughout a finite workspace.

Keywords: Translational Parallel Manipulator, Synthesis, Topology, Kinematics, Displacement

Synthèse topologique des manipulateurs parallèles

Résumé

Les méthodes de synthèse topologique des manipulateurs parallèles en translation (MPT) sont principalement basées sur la théorie des visseurs, la cinématique instantanée, et la théorie des groupes de Lie. Dans cet article, nous proposons une approche de synthèse topologique des MPTs en utilisant les équations de déplacement. Une étude des chaînes cinématiques sérielles ayant moins de 6 degrés de liberté (DDL) est d'abord effectuée afin de déduire les conditions topologiques pour celles-ci de produire des déplacements de 3 DDL en translation lorsqu'une contrainte d'orientation constante est imposée sur leurs effecteurs. Les manipulateurs parallèles composés de ces chaînes cinématiques sérielles sont en suite étudiés afin de savoir si et sous quelles conditions l'effecteur peut avoir une orientation constante.

Mots-clé: manipulateur parallèle en translation, synthèse, topologie, cinématique, déplacement

1 INTRODUCTION

From the kinematic point of view, a mechanism is a kinematic chain with one of its links specified as the base and another one as the end-effector (EE); a manipulator is a mechanism with all or some of its joints actuated; driven by the actuated joints, the EE and all other links undergo constrained motions with respect to the base. A parallel manipulator (PM) is a closed-loop mechanism in which the EE is connected to the base through at least two independent kinematic chains (subchain). A fully parallel manipulator is a PM with an n-degree-of-freedom (DOF) EE connected to the base by n independent kinematic chains, each having a single actuated joint.

Due to the closed-loop nature, PMs possess kinematic properties which are complementary to those of traditional serial manipulators (SM). Applications of PMs can be found in motion simulators, high-precision surgical tools, precision assembly tools, machine tools, and a number of industrial equipments because of their high load-carrying capacity, accurate positioning, high speed, and high capacity of acceleration.

Although PMs have been very successful in some applications, offering high performance, they are not yet completely accepted in some industrial areas, *e.g.* the machine-tool industry [1]. Complex kinematic model and limited workspace inherent to closed-form mechanisms may explain this scenario at theoretical level [2]. To overcome some of the drawbacks, one of the strategies is to connect in series two PMs of 3 DOF (the two together producing the 6-DOF mobility of the EE) in the aim to improve overall performances and make the design easier [3]. The advantages of this kind of hybrid manipulators are illustrated by a hybrid kinematic machine [4]. Therefore, the synthesis of PMs of 3 DOF has become an important design issue.

During the last two decades, a great number of novel designs of PMs have been reported in the literature and enormous effort has been devoted to their kinematic studies. Amongst the representative architectures of early translational PMs (TPM), we can cite Delta PM [5], Y-Star PM [6], Orthoglide PM [7], and 3-UPU PM [8]. Research works on these TPMs were carried out mainly on a case-by-case basis and they have little in common on synthesis and analysis methodology. As systematic synthesis approaches were gradually introduced into the topological and geometric synthesis, the number of new designs had been increasing quadratically. Based on Group Theory, the synthesis of a family of TPMs was realized by [9]. The application of group theory also leads to the synthesis of a set of spherical PMs [10]. Graph theory, which has been successfully used to planar mechanism synthesis, was used to enumerate some PMs [11]. Applying Screw Theory to the synthesis of PMs was investigated and a detailed procedure was proposed in [12]. With this approach and similar methods, a large number of topologies for TPMs were generated [13, 14, 15, 16]. Synthesis based on instantaneous kinematics was proposed in [17]. The main drawback of the methods based on Screw Theory or instantaneous kinematics is that the motion type issue for the entire workspace can not be properly addressed.

The objective of this work is to derived topology conditions for fully PMs to have only translational degrees of freedom within a finite workspace. This is done by using directly the finite displacement equations.

2 KINEMATIC MODELLING AND DEFINITIONS

The *kinematic composition* of a manipulator is the essential information about the number of its links, which link is connected to which other links by what types of joints and which joints are actuated. The *characteristic constraints* are the minimum conditions for a manipulator of a given kinematic composition to have the required kinematic properties. The *topology* of a manipulator is its kinematic composition plus the characteristic constraints. The *geometry* of a manipulator is a set of constraints on the relative locations of its joints carried by the same link and the relative locations of its links coupled by the same joint. The geometry is unique to each of the manipulators of the same topology. Joint variables of a joint describe the relative position and orientation of two links coupled by the joint. The number of joint variables is the DOF of the joint.

To simplify the kinematic parametrization and without loss of generality, joints of more than one DOF are decomposed into the combinations of 1-DOF joints. Since the EE of a PM is connected to the base by independent serial kinematic chains, the Denavit-Hartenberg notation [18] for serial mechanism can therefore be used here for each so-called subchain of PM. The links of serial chain j are identified by $(j,0)$ to (j,m_j), with $(j,0)$ for the base and (j,m_j) for the end link. A reference frame is attached to each link and is identified in the same way as the links. Since the EE and the base of a PM composed of n subchains each carry n joints, $n+1$ reference frames are defined on each of them in order to defined their geometry; $\mathcal{F}_{1,\,0} \sim \mathcal{F}_{n,\,0}$ and $\mathcal{F}_b$ are defined on the base with their z-axes aligned while $\mathcal{F}_{1,\,m_1} \sim \mathcal{F}_{n,\,m_n}$ and $\mathcal{F}_e$ are defined on the EE with their z-axes aligned. Symbols used to formulate the kinematic model are as follows:

- b, e : subscripts to identify the base and the EE;
- $\mathcal{F}_i$: reference frame attached to $link\ i$;
- $\mathbf{Q}_c$: 3×3 orientation matrix of $\mathcal{F}_c$ with respect to $\mathcal{F}_b$;
- $\mathbf{R_z}\,(\theta)$, $\mathbf{R_x}\,(\alpha)$: rotation matrices around z and x-axes by θ and α respectively;
- $\mathbf{R_{hz}}\,(\theta)$, $\mathbf{R_{hx}}\,(\alpha)$: homogeneous transformation of $\mathbf{R_z}\,(\theta)$, $\mathbf{R_x}\,(\alpha)$;
- $\mathbf{B_z}\,(d)$, $\mathbf{B_x}\,(a)$: homogeneous translation of d along z and x axis;
- $\mathbf{C}_i$: 4×4 homogeneous transformation matrix of $\mathcal{F}_i$ in $\mathcal{F}_{i-1}$;
- $\mathbf{H}_i$: 4×4 homogeneous transformation matrix of $\mathcal{F}_i$ in $\mathcal{F}_b$;
- $\mathbf{e}_i$: the k^{th} canonical vector $\boldsymbol{e}_k \equiv \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{k-1} & 1 & \underbrace{0 \ \cdots \ 0}_{n-k} \end{bmatrix}^T$

whose dimension is implicit and depends on the context.

The sequence of links in a serial chain has a corresponding sequence of homogeneous transformations that defines the position and orientation of each link relative to its neighbor in the chain.

The position and orientation of the EE of a PM is therefore defined by the product of these transformations through any serial chain, *i.e.*,

$$
\begin{aligned}
\mathbf{H}_e \;&=\; \left(\prod_{i=0}^{m_j} \mathbf{C}_{j,\,i}\right) \mathbf{C}_{j,\,e},\; j \;=\; 1,\; \cdots\; n \\
\mathbf{C}_{j,\,i} \;&=\; \mathbf{B_z}(d_{j,\,i})\mathbf{R_{hz}}(\theta_{j,\,i})\mathbf{R_{hx}}(\alpha_{j,\,i})\mathbf{B_x}(a_{j,\,i}), \quad \mathbf{C}_{j,\,e} \;= \mathbf{B_z}(d_{j,\,e})\mathbf{R_{hz}}(\theta_{j,\,e})
\end{aligned}
\tag{1}
$$

For revolute joint, $\theta_{j,\,i}$ is the joint variable while $d_{j,\,i}$ is the joint variable for prismatic joint (except for $\theta_{j,\,0}$, $d_{j,\,0}$, $\theta_{j,\,e}$, and $d_{j,\,e}$which are geometric parameters for the base and the EE). Other parameters define the geometry of the PM. As opposed to synthesis methods based on screw theory or instantenous kinematics, our synthesis method is based on finite displacement eq(1).

3 Possible subchain topologies for TPMs

By taking the orientation part of equation (1) for subchains of less than 6 joints, we get the orientation space

$$
\{\mathbf{Q}_e \mid \mathbf{Q}_e = \left[\prod_{i=0}^{m} [\mathbf{R_z}(\theta_i)\mathbf{R_x}(\alpha_i)]\right] \mathbf{R_z}(\theta_e)\},\; m < 6 \tag{2}
$$

where the subchain-identifying subscript is dropped off for simplicity. When imposing a constant orientation constraint to the EE, we get the following structure equations

$$
\mathbf{H}_e \;=\; \left(\prod_{i=0}^{m} \mathbf{C}_i\right) \mathbf{C}_e,\; m < 6 \quad \left[\prod_{i=0}^{m} [\mathbf{R_z}(\theta_i)\mathbf{R_x}(\alpha_i)]\right] \mathbf{R_z}(\theta_e) = \mathbf{Q}_0, \quad \mathbf{Q}_0 \in \{\mathbf{Q}_e\} \tag{3}
$$

where eq.(3) is equivalent to 3 scalar equations and reduces the DOF of the subchain by $m_R \leq 3$. In order to produce 3-DOF translation under eq.(3), the subchains must have

$$
m \;-\; m_R \;\geq\; 3,\; m < 6 \;\Rightarrow\; m_R \in \{0,\; 1,\; 2\} \tag{4}
$$

Since prismatic joint variables are not involved in eq.(3), m_R depends only on the number of revolute joints and their relative orientations. The spatial arrangement of revolute joints can therefore be derived. Let the revolute joints of a subchain be denoted with R_1 to R_M, then if all revolute joints are parallel eq.(3) can be written as

$$
\mathbf{F}_1\mathbf{R_z}(\sum_{i=1}^{M} \theta_{R_i})\mathbf{F}_2 = \mathbf{Q}_0 \tag{5}
$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are constant matrices determined by the subchain geometry. Equation (5) means that the DOF of the subchain is reduced by 1, *i.e.* $m_R = 1$.

If R_k and R_{k+1} with $0 < k < M$ are not parallel then eq.(3) can be written as

$$
\mathbf{F}_3(\theta_{R_1}, \ldots, \theta_{R_{k-1}})\mathbf{R_z}(\theta_{R_k})\mathbf{F}_4\mathbf{R_z}(\theta_{R_{k+1}})\mathbf{F}_5(\theta_{R_{k+2}}, \ldots, \theta_{R_M}) = \mathbf{Q}_0 \tag{6}
$$

where $\mathbf{F}_3$ and $\mathbf{F}_5$ are known functions while $\mathbf{F}_4$ is a constant orthogonal matrix. From eq.(6) and the fact that $\mathbf{R_z}(\theta)\boldsymbol{e}_3 = \boldsymbol{e}_3$ and $\boldsymbol{e}_3^T\mathbf{R_z}(\theta) = \boldsymbol{e}_3^T$, we get

$$\begin{aligned}
\mathbf{R_z}(\theta_{R_k})\mathbf{F}_4\boldsymbol{e}_3 &= \mathbf{F}_3(\theta_{R_1},\ldots,\theta_{R_{k-1}})^T\mathbf{Q}_0\mathbf{F}_5(\theta_{R_{k+2}},\ldots,\theta_{R_M})^T\boldsymbol{e}_3 && (7)\\
\boldsymbol{e}_3^T\mathbf{F}_4\mathbf{R_z}(\theta_{R_{k+1}}) &= \boldsymbol{e}_3^T\mathbf{F}_3(\theta_{R_1},\ldots,\theta_{R_{k-1}})^T\mathbf{Q}_0\mathbf{F}_5(\theta_{R_{k+2}},\ldots,\theta_{R_M})^T && (8)
\end{aligned}$$

which mean that θ_{R_k} and $\theta_{R_{k+1}}$ can be solved as function of other joint variables, the DOF of the subchain is reduced by 2, *i.e.* $m_R = 2$. It can therefore be concluded that to satisfy eq.(4): 1) 3-joint subchains can only have prismatic joints; 2) all revolute joints of 4-joint subchains should be parallel. Now we further analyze subchains in which not all revolute joints are parallel. If 1) R_1 to R_k are parallel; 2) R_k and R_{k+1} are not parallel; 3) R_{k+1} to R_l are parallel; 4) R_l to R_{l+1} are not parallel; 5) R_{l+1} to R_M are parallel, where $0 < k < l < M$, then eq.(3) can be written as

$$\mathbf{F}_6\mathbf{R_z}(\sum_{i=1}^{k}\theta_{R_i})\mathbf{F}_7\mathbf{R_z}(\sum_{i=k+1}^{l}\theta_{R_i})\mathbf{F}_8\mathbf{R_z}(\sum_{i=l+1}^{M}\theta_{R_i})\mathbf{F}_9 = \mathbf{Q}_0 \tag{9}$$

where $\mathbf{F}_6 \sim \mathbf{F}_9$ are constant orthogonal matrices. Equation (9) is equivalent to 3 scaler equations with $\sum_{i=1}^{k}\theta_{R_i}$, $\sum_{i=k+1}^{l}\theta_{R_i}$, and $\sum_{i=l+1}^{M}\theta_{R_i}$ as unknowns. Performing linear transformations on eq.(9), we get

$$\begin{aligned}
\boldsymbol{e}_3^T\mathbf{F}_7\mathbf{R_z}(\sum_{i=k+1}^{l}\theta_{R_i})\mathbf{F}_8\boldsymbol{e}_3 &= \boldsymbol{e}_3^T\mathbf{F}_6^T\mathbf{Q}_0\mathbf{F}_9^T\boldsymbol{e}_3 && (10)\\
\mathbf{R_z}(\sum_{i=1}^{k}\theta_{R_i})\mathbf{F}_7\mathbf{R_z}(\sum_{i=k+1}^{l}\theta_{R_i})\mathbf{F}_8\boldsymbol{e}_3 &= \mathbf{F}_6^T\mathbf{Q}_0\mathbf{F}_9^T\boldsymbol{e}_3 && (11)\\
\boldsymbol{e}_3^T\mathbf{F}_7\mathbf{R_z}(\sum_{i=k+1}^{l}\theta_{R_i})\mathbf{F}_8\mathbf{R_z}(\sum_{i=l+1}^{M}\theta_{R_i}) &= \boldsymbol{e}_3^T\mathbf{F}_6^T\mathbf{Q}_0\mathbf{F}_9^T && (12)
\end{aligned}$$

$\sum_{i=k+1}^{l}\theta_{R_i}$ can be solved from eq.(10) and in general case, $\sum_{i=1}^{k}\theta_{R_i}$ and $\sum_{i=l+1}^{M}\theta_{R_i}$ can then be solved from eqs.(11) and (12) respectively, reducing the DOF of the subchain by 3, *i.e.* $m_R = 3$. However, if the subchain has such a geometry that

$$\begin{aligned}
\forall\theta_S \in \{\sum_{i=k+1}^{l}\theta_{R_i} \mid \boldsymbol{e}_3^T\mathbf{F}_7\mathbf{R_z}(\sum_{i=k+1}^{l}\theta_{R_i})\mathbf{F}_8\boldsymbol{e}_3 &= \boldsymbol{e}_3^T\mathbf{F}_6^T\mathbf{Q}_0\mathbf{F}_9^T\boldsymbol{e}_3\} \quad \exists\theta_C \in \mathbf{R}\\
\mathbf{F}_7\mathbf{R_z}(\theta_S)\mathbf{F}_8 = \mathbf{R_z}(\theta_C) & && (13)
\end{aligned}$$

then $\sum_{i=1}^{k}\theta_{R_i}$ and $\sum_{i=l+1}^{M}\theta_{R_i}$ vanish from eqs. (11) and eq.(12) respectively. In this case, from eqs.(9) and (13) we get

$$\mathbf{F}_6\mathbf{R_z}(\sum_{i=1}^{k}\theta_{R_i} + \sum_{i=l+1}^{M}\theta_{R_i} + \theta_C)\mathbf{F}_9 = \mathbf{Q}_0 \tag{14}$$

and the DOF of the subchain is actually reduced by 2 (eq.10 and eq.14), *i.e.* $m_R = 2$. The physical interpretation of eqs.(13) and (14) is that $\mathbf{Q}_0$ is attained by the EE while axes of R_k and R_{l+1} are parallel. In order for the axes of R_k and R_{l+1} to reach the parallel relative location, the angle between R_k and R_{k+1} should be equal to the angle between R_l and R_{l+1}.

Based on the revolute joint situations, subchains can be classified into the following categories:

T-subchain: a subchain with only prismatic joints, non of them being parallel;

I-subchain: a subchain whose revolute joints are all parallel;

A-subchain: a subchain where only one pair of adjacent revolute joints are not parallel;

Z-subchain: a subchain where only two pairs of adjacent revolute joints are not parallel and the two pairs of the non parallel revolute joints have the same angle.;

Y-subchain: a subchain which is not a Z-subchain and has more than one pair of adjacent revolute joints which are not parallel.

Then, from the above analysis, the possible subchain topologies for TPMs are derived as listed in table 1.

Table 1: Possible subchain topologies for TPMs

	3-joint	4-joint	5-joint	6-joint
T-subchain	Yes	No	No	No
I-subchain	No	Yes	No	No
A-subchain	No	No	Yes	No
Z-subchain	No	No	Yes	Yes
Y-subchain	No	No	No	Yes

4 Analysis of the EE orientation space of PMs of 3 DOF

The orientation space of a PM is formed by the orientation part of the set of solutions of eq.(1). In order to synthesize topologies of TPMs, the reasoning is as follows: if

1. the orientation part of the set of solutions of eq. (1) has only finite number of elements, (*i.e.* all solitary subspaces of the orientation space is a single element subspace, no orientation path exists between any two of them, meaning that the EE of the PM can not pass from one orientation to another without being reassembled), and

2. each subchain allows the EE to have 3 DOFs in translation without changing its orientation,

then the PM is naturally a TPM.

4.1 PMs with a T-subchain

Under the constraint of the T-subchain, the orientation space of the EE is a single element space determined by the geometry of this subchain, *i.e.*

$$\mathbf{Q}_e = \left[\prod_{i=0}^{m_1} [\mathbf{R_z}(\theta_{1,\,i})\mathbf{R_x}(\alpha_{1,\,i})]\right] \mathbf{R_z}(\theta_{1,\,e}),\ m_1 = 3 \tag{15}$$

The EE will not have any displacement in orientation. From the point of view of eliminating displacements in orientation, the second and third subchains can be of any possible subchain topologies derived in the previous section (table 1). However, if the dimension of the orientation space of any of the other subchains is lower then 3, the PM becomes overconstrained, exact geometries of the base and the EE are necessary for eq.(1) to have any real solution, *i.e.* for the PM to be assembled.

4.2 PMs with an I-subchain

Let the first subchain be an I-subchain, then from eq. (5), we get

$$\mathbf{F}_{1,\,1}\mathbf{R_z}(\sum_{i=1}^{M_1} \theta_{R_{1,\,i}})\mathbf{F}_{1,\,2} = \mathbf{Q}_e \tag{16}$$

If the second subchain is also an I-subchain, then

$$\mathbf{F}_{2,\,1}\mathbf{R_z}(\sum_{i=1}^{M_2} \theta_{R_{2,\,i}})\mathbf{F}_{2,\,2} = \mathbf{Q}_e \tag{17}$$

Combining eqs.(16) and (17) yields

$$\mathbf{F}^T_{2,\,1}\mathbf{F}_{1,\,1}\mathbf{R_z}(\sum_{i=1}^{M_1} \theta_{R_{1,\,i}})\mathbf{F}_{1,\,2}\mathbf{F}^T_{2,\,2} = \mathbf{R_z}(\sum_{i=1}^{M_2} \theta_{R_{2,\,i}}) \tag{18}$$

Since $\mathbf{R_z}(\theta)\boldsymbol{e}_3 = \boldsymbol{e}_3$, from eq.(18) we have

$$\mathbf{F}^T_{2,\,1}\mathbf{F}_{1,\,1}\mathbf{R_z}(\sum_{i=1}^{M_1} \theta_{R_{1,\,i}})\mathbf{F}_{1,\,2}\mathbf{F}^T_{2,\,2}\boldsymbol{e}_3 = \boldsymbol{e}_3 \tag{19}$$

It is obvious that eq.(16) and (19) define a single element orientation space. If the second subchain is an A-subchain and the non parallel revolute joints are R_{2,k_2} and R_{2,k_2+1}, then

$$\mathbf{F}_{2,3}\mathbf{R_z}(\sum_{i=1}^{k_2} \theta_{R_{2,i}})\mathbf{F}_{2,4}\mathbf{R_z}(\sum_{i=k_2+1}^{M_2} \theta_{R_{2,i}})\mathbf{F}_{2,5} = \mathbf{Q}_e \tag{20}$$

Combining eqs.(16) and (20) leads to

$$\mathbf{R_z}(\sum_{i=1}^{k_2} \theta_{R_{2,i}})\mathbf{F}_{2,4}\mathbf{R_z}(\sum_{i=k_2+1}^{M_2} \theta_{R_{2,i}})\mathbf{F}_{2,5}\mathbf{F}^T_{1,\,2} = \mathbf{F}^T_{2,3}\mathbf{F}_{1,\,1}\mathbf{R_z}(\sum_{i=1}^{M_1} \theta_{R_{1,\,i}}) \tag{21}$$

Eliminating $\sum_{i=1}^{k_2}\theta_{R_{2,i}}$ and $\sum_{i=1}^{M_1}\theta_{R_{1,\ i}}$ from eq.(21), we get

$$\boldsymbol{e}_3^T\mathbf{F}_{2,4}\mathbf{R}_{\mathbf{z}}(\sum_{i=k_2+1}^{M_2}\theta_{R_{2,i}})\mathbf{F}_{2,5}\mathbf{F}_{1,\ 2}^T\boldsymbol{e}_3=\boldsymbol{e}_3^T\mathbf{F}_{2,3}^T\mathbf{F}_{1,\ 1}\boldsymbol{e}_3 \tag{22}$$

where $\sum_{i=k_2+1}^{M_2}\theta_{R_{2,i}}$ is the only unknown and can have at most two solutions. It is clear that under the constraints of an I-subchain and an A-subchain, the EE has only two possible orientations, each corresponding to an assembly mode; given an assembly mode, it is impossible for the EE to change orientation. The third subchain can be of any topologies listed in table 1.

4.3 PMs with three A-subchains

If the three subchains are all A-subchains, we have

$$\mathbf{F}_{1,3}\mathbf{R}_{\mathbf{z}}(\sum_{i=1}^{k_1}\theta_{R_{1,i}})\mathbf{F}_{1,4}\mathbf{R}_{\mathbf{z}}(\sum_{i=k_1+1}^{M_1}\theta_{R_{1,i}})\mathbf{F}_{1,5}=\mathbf{Q}_e \tag{23}$$

$$\mathbf{F}_{2,3}\mathbf{R}_{\mathbf{z}}(\sum_{i=1}^{k_2}\theta_{R_{2,i}})\mathbf{F}_{2,4}\mathbf{R}_{\mathbf{z}}(\sum_{i=k_2+1}^{M_2}\theta_{R_{2,i}})\mathbf{F}_{2,5}=\mathbf{Q}_e \tag{24}$$

$$\mathbf{F}_{3,3}\mathbf{R}_{\mathbf{z}}(\sum_{i=1}^{k_3}\theta_{R_{3,i}})\mathbf{F}_{3,4}\mathbf{R}_{\mathbf{z}}(\sum_{i=k_3+1}^{M_3}\theta_{R_{3,i}})\mathbf{F}_{3,5}=\mathbf{Q}_e \tag{25}$$

let

$$\phi_{j,1}\equiv\sum_{i=1}^{k_j}\theta_{R_{j,i}},\ \phi_{j,2}\equiv\sum_{i=k_j+1}^{M_j}\theta_{R_{j,i}},\ j=1,\ 2,\ 3$$

then from eqs.(23) to (25) and upon rearrangement, we have

$$\mathbf{F}_{2,3}^T\mathbf{F}_{1,3}\mathbf{R}_{\mathbf{z}}(\phi_{1,1})\mathbf{F}_{1,4}\mathbf{R}_{\mathbf{z}}(\phi_{1,2})\mathbf{F}_{1,5}\mathbf{F}_{2,5}^T=\mathbf{R}_{\mathbf{z}}(\phi_{2,1})\mathbf{F}_{2,4}\mathbf{R}_{\mathbf{z}}(\phi_{2,2}) \tag{26}$$

$$\mathbf{F}_{3,3}^T\mathbf{F}_{1,3}\mathbf{R}_{\mathbf{z}}(\phi_{1,1})\mathbf{F}_{1,4}\mathbf{R}_{\mathbf{z}}(\phi_{1,2})\mathbf{F}_{1,5}\mathbf{F}_{3,5}^T=\mathbf{R}_{\mathbf{z}}(\phi_{3,1})\mathbf{F}_{3,4}\mathbf{R}_{\mathbf{z}}(\phi_{3,2}) \tag{27}$$

Eliminating the unknowns on the right sides of eqs.(26) and (27), we have

$$\boldsymbol{e}_3^T\mathbf{F}_{2,3}^T\mathbf{F}_{1,3}\mathbf{R}_{\mathbf{z}}(\phi_{1,1})\mathbf{F}_{1,4}\mathbf{R}_{\mathbf{z}}(\phi_{1,2})\mathbf{F}_{1,5}\mathbf{F}_{2,5}^T\boldsymbol{e}_3=\boldsymbol{e}_3^T\mathbf{F}_{2,4}\boldsymbol{e}_3 \tag{28}$$

$$\boldsymbol{e}_3^T\mathbf{F}_{3,3}^T\mathbf{F}_{1,3}\mathbf{R}_{\mathbf{z}}(\phi_{1,1})\mathbf{F}_{1,4}\mathbf{R}_{\mathbf{z}}(\phi_{1,2})\mathbf{F}_{1,5}\mathbf{F}_{3,5}^T\boldsymbol{e}_3=\boldsymbol{e}_3^T\mathbf{F}_{3,4}\boldsymbol{e}_3 \tag{29}$$

Let

$$sin(\phi_{1,1})\equiv\frac{2s}{1+s^2},\quad cos(\phi_{1,1})\equiv\frac{1-s^2}{1+s^2},\quad sin(\phi_{1,2})\equiv\frac{2t}{1+t^2},\quad cos(\phi_{1,2})\equiv\frac{1-t^2}{1+t^2}$$

then the following set of equations can be derived from eqs.(28) and (29):

$$[\,t^2\ \ t\ \ 1\,]\mathbf{A}[\,s^2\ \ s\ \ 1\,]^T\ =\ \mathbf{0} \tag{30}$$

where **A** is a 3-by-3 matrix whose elements depend only on the geometry of the subchains.

Equation (30) is ready to be solved and can have at most 8 real solutions. That is to say under the constraints of three A-subchains the EE can have only finite number of orientations with each orientation corresponding to an assembly mode. Given an assembly mode, it is impossible for the EE to change orientation.

4.4 PMs with one Z-subchain and two A-subchains

Let the first subchain be the Z-subchain and the rest be the A-subchains. The orientation equations are

$$\mathbf{F}_{1,6}\mathbf{R_z}(\sum_{i=1}^{k_1}\theta_{R_{1,i}})\mathbf{F}_{1,7}\mathbf{R_z}(\sum_{i=k_1+1}^{l_1}\theta_{R_{1,i}})\mathbf{F}_{1,8}\mathbf{R_z}(\sum_{i=l_1+1}^{M_1}\theta_{R_{1,i}})\mathbf{F}_{1,9}=\mathbf{Q}_e \tag{31}$$

$$\mathbf{F}_{2,3}\mathbf{R_z}(\sum_{i=1}^{k_2}\theta_{R_{2,i}})\mathbf{F}_{2,4}\mathbf{R_z}(\sum_{i=k_2+1}^{M_2}\theta_{R_{2,i}})\mathbf{F}_{2,5}=\mathbf{Q}_e \tag{32}$$

$$\mathbf{F}_{3,3}\mathbf{R_z}(\sum_{i=1}^{k_3}\theta_{R_{3,i}})\mathbf{F}_{3,4}\mathbf{R_z}(\sum_{i=k_3+1}^{M_3}\theta_{R_{3,i}})\mathbf{F}_{3,5}=\mathbf{Q}_e \tag{33}$$

let

$$\begin{aligned}&\phi_{1,1}\equiv\sum_{i=1}^{k_1}\theta_{R_{1,i}},\ \phi_{1,2}\equiv\sum_{i=k_1+1}^{l_1}\theta_{R_{1,i}},\ \phi_{1,3}\equiv\sum_{i=l_1+1}^{M_1}\theta_{R_{1,i}},\\&\phi_{j,1}\equiv\sum_{i=1}^{k_j}\theta_{R_{j,i}},\ \phi_{j,2}\equiv\sum_{i=k_j+1}^{M_j}\theta_{R_{j,i}},\ j=2,3\end{aligned} \tag{34}$$

then from eqs.(31) to (33), we have

$$\mathbf{F}_{2,3}^T\mathbf{F}_{1,6}\mathbf{R_z}(\phi_{1,1})\mathbf{F}_{1,7}\mathbf{R_z}(\phi_{1,2})\mathbf{F}_{1,8}\mathbf{R_z}(\phi_{1,3})\mathbf{F}_{1,9}\mathbf{F}_{2,5}^T=\mathbf{R_z}(\phi_{2,1})\mathbf{F}_{2,4}\mathbf{R_z}(\phi_{2,2}) \tag{35}$$

$$\mathbf{F}_{3,3}^T\mathbf{F}_{1,6}\mathbf{R_z}(\phi_{1,1})\mathbf{F}_{1,7}\mathbf{R_z}(\phi_{1,2})\mathbf{F}_{1,8}\mathbf{R_z}(\phi_{1,3})\mathbf{F}_{1,9}\mathbf{F}_{3,5}^T=\mathbf{R_z}(\phi_{3,1})\mathbf{F}_{3,4}\mathbf{R_z}(\phi_{3,2}) \tag{36}$$

From the Z-subchain property (eq.13), we know that there exists θ_S such that

$$\phi_{1,2}=\theta_S,\quad \exists\theta_C\in\mathbf{R},\quad \mathbf{F}_{1,7}\mathbf{R_z}(\theta_S)\mathbf{F}_{1,8}=\mathbf{R_z}(\theta_C) \tag{37}$$

When $\phi_{1,2}=\theta_S$, eqs.(35) and (36) become

$$\mathbf{F}_{2,3}^T\mathbf{F}_{1,6}\mathbf{R_z}(\phi_{1,1}+\phi_{1,3}+\theta_C)\mathbf{F}_{1,9}\mathbf{F}_{2,5}^T=\mathbf{R_z}(\phi_{2,1})\mathbf{F}_{2,4}\mathbf{R_z}(\phi_{2,2}) \tag{38}$$

$$\mathbf{F}_{3,3}^T\mathbf{F}_{1,6}\mathbf{R_z}(\phi_{1,1}+\phi_{1,3}+\theta_C)\mathbf{F}_{1,9}\mathbf{F}_{3,5}^T=\mathbf{R_z}(\phi_{3,1})\mathbf{F}_{3,4}\mathbf{R_z}(\phi_{3,2}) \tag{39}$$

Elimination of $\phi_{2,1}$ and $\phi_{2,2}$ from eq.(38) yields

$$\boldsymbol{e}_3^T\mathbf{F}_{2,3}^T\mathbf{F}_{1,6}\mathbf{R_z}(\phi_{1,1}+\phi_{1,3}+\theta_C)\mathbf{F}_{1,9}\mathbf{F}_{2,5}^T\boldsymbol{e}_3=\boldsymbol{e}_3^T\mathbf{F}_{2,4}\boldsymbol{e}_3 \tag{40}$$

$\phi_{1,1}+\phi_{1,3}$ can be easily solved as $\phi_{1,1}+\phi_{1,3}=C,\ C\in\mathbf{R}$. When $\phi_{1,2}\neq\theta_S$, for any given $\phi_{1,3}$, $\phi_{1,1}$ and $\phi_{1,2}$ can be solved from eqs.(35) and (36) in the same way as eqs.(26) and (27). Hence, $\phi_{1,1}=f_1(\phi_{1,3}),\ \phi_{1,2}=f_1(\phi_{1,3}),\ \forall\phi_{1,3}\in\mathbf{R},\ \phi_{1,2}\neq\theta_S$ From eq.(31) we know that the orientation of the EE is determined by the triple $[\phi_{1,1},\ \phi_{1,2},\ \phi_{1,3}]$ and the orientation space is a metric space $X=\{[\phi_{1,1},\ \phi_{1,2},\ \phi_{1,3}]\}$ which is the union of two subspaces:

$$\begin{aligned} X &= X_1\cup X_2, \quad X_1=\{[\phi_{1,1},\phi_{1,2},\phi_{1,3}]\mid \phi_{1,2}=\theta_S;\phi_{1,1}+\phi_{1,3}=C;\theta_S,C\in\mathbf{R}\} \\ X_2 &= \{[\phi_{1,1},\phi_{1,2},\phi_{1,3}]\mid \phi_{1,2}\neq\theta_S;\phi_{1,1}=f_1(\phi_{1,3});\phi_{1,2}=f_2(\phi_{1,3});\theta_S,\forall\phi_{1,3}\in\mathbf{R}\} \end{aligned}$$

From eqs. (31)and (37), it can be derived that

$$\mathbf{F}_{1,6}\mathbf{R}_{\mathbf{z}}(C+\theta_C)\mathbf{F}_{1,9}=\mathbf{Q}_e$$

which means that the EE has a constant orientation within subspace X_1. Since X_1 and X_2 are of one dimension, there exists a neighborhood $B_\varepsilon(\boldsymbol{x})$ within X_1 such that $B_\varepsilon(\boldsymbol{x})\cap X_2=NULL$ That is to say it is impossible for the EE to change orientation within $B_\varepsilon(\boldsymbol{x})$. With the same procedure, it can be proven that such a neighborhood exists for PMs with two or three Z-subchains.

5 Topological synthesis of TPMs

With the analyses carried out in the previous sections, the topological synthesis of TPMs becomes easier and can be summarized as follows:

The first step is to determine the type of each subchain. The type of the first subchain can be of any of those listed in table 1. Depending on the choice of the first subchain, the second and third can be determined such that a constant orientation configuration neighborhood exists.

1. If the first subchain is a T-subchain, then the second and the third can be any of those listed in table 1;

2. If the first subchain is an I-subchain then at least one of the second and the third should not be an Y-subchain;

3. If the first subchain is an A-subchain or Z-subchain, then the second and the third can be either an A-subchain or a Z-subchain.

This can also serve as a verification for the synthesis based on instantaneous kinematics. Then, the next step is to determine the topology of a subchain of a given type so as to generate 3-DOF translation. This can be done by using the synthesis methods for serial kinematic chains. This topological synthesis approach can be illustrated by existing TPM topologies. If the first subchain is an I-subchain then the second and third subchains can also be an I-subchain in order to synthesize a TPM. Using the synthesis methods for serial kinematic chains, one knows that a $P_1R_1R_1R_1$ (P denotes prismatic joint and R for revolute joint; joints with the same subscript are parallel) subchain is an I-subchain and possesses 3 DOF when its EE is constrained to a constant orientation. Fig. (1a) shows a TPM of this topology. If the first subchain is an A-subchain then the second and the third subchains can also be an A-subchain. We know that a $R_1R_1R_2R_2R_2$ subchain is an A-subchain and satisfies the condition for forming a TPM. Fig. (1b) shows a TPM with 3 $R_1R_1R_2R_2R_2$ subchains. Fig. (1c) shows a TPM with 3 $R_1R_1R_2R_2R_1$ subchains which are Z-subchains.

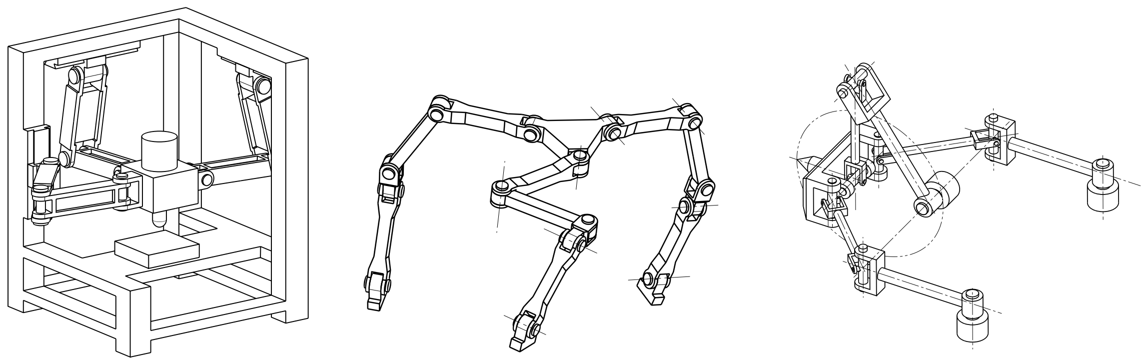

Figure 1: TPMs with a) I-subchains [19]; b) A-subchains [20]; c)Z-subchains [15]

6 CONCLUSION

The proposed kinematic model applies to the most general topologies and geometries of 3-DOF PMs and therefore allows a thorough analysis of how the translational displacement and the configuration of a subchain are affected by a constant orientation constraint on the EE. The topological constraints are derived for a serial kinematic chain of less than 6 joints to produce 3-DOF translation with a constant orientation of the EE. Subchains can be classified as T-subchain, I-subchain, A-subchain, Z-subchain, and Y-subchain. The analysis of the orientation solutions of the forward kinematics of all subchain combinations confirms which kinds of combination can produce a constant EE orientation in a finite configuration space. The finite configuration space may be the entire configuration space, a particular assembly mode, or a neighborhood in the configuration space.

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