Controllable Workspace of General Cable Driven Redundant Parallel Manipulator Based on Fundamental Wrench

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Abstract

Workspace analysis is one of the most important issues in robotic manipulator design. This paper introduces a set of newly defined fundamental wrenches that opens new horizons for physical interpretation of controllable workspace of cable driven redundant parallel manipulators. Moreover, an analytical method is proposed to specify the controllable workspace of general redundant cable-driven parallel manipulators based on this set of fundamental wrenches. In the proposed method, an analytic approach based on linear algebra is employed to derive the boundary of controllable workspace. Finally, the proposed method is illustrated through spatial example.

Keywords: fundamental wrench, controllable workspace, cable driven redundant parallel manipulator.

1 INTRODUCTION

Cable-driven redundant parallel manipulators (CDRPM) has several attractive features and some advantage compared to conventional parallel manipulators. In spite of many advantages and promising potentials, there are many challenging problems in the design and development of cable robot. Workspace analysis is always a crucial issue in the design of any robot, as in cable manipulator. The uni-directional constraint imposed by cables causes the kind of analysis more difficult than traditional robotic manipulator. In literature, four different types of workspace have been introduced based on various definitions. A number of researchers addressed the set of postures that the end-effector can attain statically while only taking gravity into account [1], [2]. Some researchers addressed a set of postures when cable robots are needed to exert specific wrench to interact with environment besides maintaining its own static equilibrium. Ebert-Uphoff and Voglewede described this type of workspace as wrench feasible workspace in [3]. Another workspace that has been introduced is the dynamic workspace along with a set of wrench called pseudo-pyramid, defined by Barrette and Gosselin as the set of all postures of the end-effector of the cable robot with specific acceleration requirement[4]. Finally, one of the most general workspace definitions is referred to the workspace in which any wrench can be generated at the moving platform while cables are in tension. Verhoeven and Hiller term such workspace as controllable workspace [5]. This kind of workspace depends only on geometry of manipulator such as position of fixed and moving attachment point [6]. Therefore, it is interested in general design view point. Pham *et al.* [7] proposed a ”recursive dimension reduction algorithm” to check the force-closure condition of cable manipulators. Although the algorithm is systematic, no mathematical proof was provided to show that it is equivalent to the original force-closure theorem . Gouttefarde and Gosselin addressed the same concept named Wrench-Closure Workspace (WCW), and determined the boundaries of the workspace for planar cable robots analytically [6]. Null space analysis is the most common methods for determination of this kind of workspace, which is associated with challenges like extreme complexity of computations and increasing of the degrees of freedom and degrees of redundancy of the robot. In this paper among the alternative notions introduced for the workspace in which any wrench can be generated at the moving platform while cables are in tension, the term *controllable workspace* is used. All the proposed method including the analytical and numerical methods proposed to analyze this problem, suffers from a lack of physical interpretation of controllable workspace.

In this paper, a set of novel external wrenches called fundamental wrenches in introduced in order to provide a physical interpretation of controllable workspace. Moreover, an analytical method is developed to determine the controllable workspace of redundant cable-driven parallel manipulators based on fundamental wrenches. The proposed method is generally applicable to any cable manipulators with any redundant cables as long as its Jacobian matrix is of full rank. In the proposed method, an analytic approach based on linear algebra is employed to derive the boundary of controllable workspace, and this method is applied to spatial case study.

The remaining of the paper is organized as follows: fundamental wrench is introduced in next section, which is followed by elaboration of the analytic approach in order to determine the controllable workspace in section 3. The implementation of the proposed method on the case study in given in section 4, and section 5 summarizes the concluding remarks.

2 FUNDAMENTAL WRENCH ANALYSIS

The general structure of a cable driven parallel manipulator is shown in figure 1. In this manipulator, the moving platform is supported by m limbs. For the CDRPM the relationship between tension force of cable and external wrench acting on the moving platform center point $\mathbf{G}$ is given by:

$$\mathbf{A}\mathbf{f} = -\mathbf{w}, \mathbf{A} = -\mathbf{J}^T \tag{1}$$

In which, $\mathbf{w}$ is the external wrench acting on the moving platform center point $\mathbf{G}$, $\mathbf{f}$ denotes the vector of cable forces, $\mathbf{A}$ denotes the structure matrix [8], and $\mathbf{J}$ is the manipulator Jacobian matrix. The Jacobian matrix of a n DOF m cable driven redundant parallel manipulator can be written as following [9]:

$$\mathbf{J} = \begin{pmatrix} \hat{\mathbf{S}}_1^T & (\mathbf{E}_1 \times \hat{\mathbf{S}}_1)^T \\ \vdots & \vdots \\ \hat{\mathbf{S}}_m^T & (\mathbf{E}_m \times \hat{\mathbf{S}}_m)^T \end{pmatrix} \tag{2}$$

In which, as illustrated in figure 1, $\hat{\mathbf{S}}_i$ and $\mathbf{E}_i$ are unit vector of i^{th} cable and position vector of the moving attachment point of the i^{th} limb measured with respect to point $\mathbf{G}$.

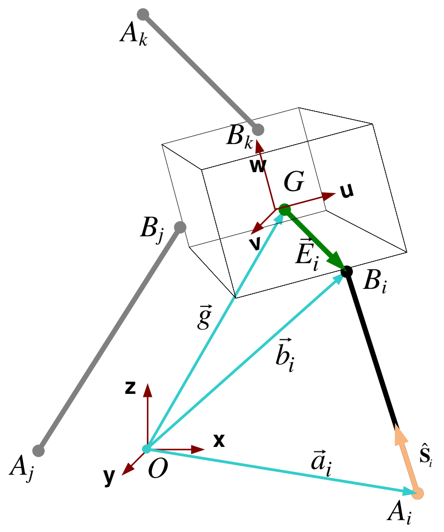


Figure 1: The general structure of a cable-driven parallel manipulator and its vectors.

In order to determine the controllable workspace of cable manipulator, the position and orientation of the manipulator through which equation (1) is solvable for nonnegative cable forces must be determined, in presence of external wrenches applied to the moving platform. Note that the CDRPM Jacobian matrix $\mathbf{J}$ is a non-square $m \times n$ matrix. Furthermore, there are infinity many solutions for $\mathbf{f}$ to be projected into $\mathbf{w}$. The simplest solution would be a minimum norm solution, which is found from the pseudo-inverse of $\mathbf{J}^T$as follows [10]:

$$\mathbf{f} = -\mathbf{A}^{\dagger}\mathbf{w} \tag{3}$$

This solution does not guarantee that the components of $\mathbf{f}$ will be nonnegative. For the linear equation (1), and the general form of least square solution is given by [10]:

$$\mathbf{f} = -\mathbf{A}^{\dagger}\mathbf{w} + N\mathbf{h} \tag{4}$$

In which, N is a matrix whose columns form a basis for the null-space of matrix $\mathbf{A}$ and $\mathbf{h}$ is a real vector whose $N\mathbf{h}$ a positive null vector in the null-space of matrix $\mathbf{A}$. However, we are seeking through the general solution According to equation (4), every solution is obtained by changing $\mathbf{h}$ to have positive tension in the cables. This solution might exist or not and ther might exist many solution to comply this constraint.

Definition I: The *feasible set* is a set of all nonnegative solution is of equation (1) derived from the general solution equation (4) that satisfies obtained by changing $\mathbf{h}$ for equation (4) that satisfy the following definition condition:

$$\mathbf{F}_{\mathbf{feas}} = \{\mathbf{f} \mid \mathbf{f} = -\mathbf{A}^{\dagger}\mathbf{w} + N\mathbf{h}, \mathbf{f} \geq \mathbf{0}\} \tag{5}$$

Definition II: the *minimum feasible solution* ($\mathbf{f}^*$) is a component of feasible set that minimizes has the minimum 2-norm of $\mathbf{f}$. Such solution can be found from the following constrained optimization problem:

$$\min ||\mathbf{f}|| \;\; subject \;\; to: \;\; \mathbf{Af} = -\mathbf{w} \;\; , \mathbf{f} \geq \mathbf{0} \tag{6}$$

A set of all manipulator configurations ($\mathbf{X}$) belongs to the Controllable workspace, if and only if the feasible set is nonempty for any wrench exerted on the end-effector of manipulator, i.e., there exist the positive vector of null space of structure matrix [5].

Let us introduce the concept of fundamental wrench in here to give a physical interpretation to the minimum feasible solution for a cable driven manipulator at a given position and orientation. A Fundamental wrenches are is defined as following, as a set of specific external wrenches that depends only on the geometry of manipulator.

Definition III: The *set of fundamental wrench* refers to a set of m vectors; each of them is equal to a opposite direction of column vector of Jacobian transpose as:

$$\mathbf{W}_F = \left[\mathbf{w}_1, ..., \mathbf{w}_m\right], \ \mathbf{w}_i = -\begin{pmatrix} \hat{\mathbf{S}}_i \\ \mathbf{E}_i \times \hat{\mathbf{S}}_i \end{pmatrix} \tag{7}$$

Note that fundamental wrench is configuration dependent. The following Lemma provides a physical interpretation of the fundamental wrench for the minimum feasible solution of a cable-driven manipulator.

Lemma I: If any wrench from the set of fundamental wrenches is applied on the moving platform center point $\mathbf{G}$, then the minimum value of feasible non-negative force in the corresponding cable will be *zero* at the given configuration.

Proof: Assume that the i^{th} vector of fundamental wrench ($\mathbf{w}_i$) is applied on the moving platform center point $\mathbf{G}$, then according to equation (1):

$$\mathbf{A}\mathbf{f} = -\mathbf{w}_i, \mathbf{A} = -\mathbf{J}^T \tag{8}$$

Following definition III, the i^{th} vector of fundamental wrench ($\mathbf{w}_i$) is the i^{th} vector of structure matrix $\mathbf{A}$.

$$\mathbf{w}_i = \mathbf{A}_i \tag{9}$$

Substitute equation (9) into equation (8):

$$\mathbf{A}\mathbf{f} = -\mathbf{A}_i \tag{10}$$

According to definition I, consider $\mathbf{F}_{feas-i}$ as a solution of the feasible set for equation (10), whose components are represented by $f_{1i}, ..., f_{mi}$. All of these components correspond to the i^{th} vector of fundamental wrench ($\mathbf{w}_i$), and satisfy equation (10). Let us write the resulting equation component-wise:

$$\mathbf{A}_1 f_{1i} + ... + \mathbf{A}_m f_{mi} = -\mathbf{A}_i \tag{11}$$

Move the right hand side component to left hand side:

$$\mathbf{A}_1 f_{1i} + ... + \mathbf{A}_i(1 + f_{ii}) + ... + \mathbf{A}_m f_{mi} = 0 \tag{12}$$

Divide All components by $(1 + f_{ii})$ whose $(1 + f_{ii})$ is strictly positive and write the resulting equation:

$$\mathbf{A}_1 \frac{f_{1i}}{(1 + f_{ii})} + ... + \mathbf{A}_i + ... + \mathbf{A}_m \frac{f_{mi}}{(1 + f_{ii})} = 0 \tag{13}$$

Move the i^{th} component to the right hand side:

$$\mathbf{A}_1\tau_{1i} + ... + \mathbf{A}_{(i-1)}\tau_{(i-1)i} + \mathbf{A}_{(i+1)}\tau_{(i+1)i} + ... + \mathbf{A}_m\tau_{mi} = -\mathbf{A}_i, \mathbf{w}_i = \mathbf{A}_i \tag{14}$$

In which, τ_{ji} denotes the normalized nonnegative force of j^{th} cablewhere the i^{th} vector of fundamental wrench ($\mathbf{w}_i$) is applied on the moving platform center point $\mathbf{G}$.

$$\tau_{ji} = \frac{f_{ji}}{(1 + f_{ii})} \tag{15}$$

Since the i^{th} term is missing in the left hand side of equation (14), this equation can be interpreted as the case that by exerting i^{th} fundamental wrench to the end effector, no projection of such wrench is seen on the corresponding i^{th} cable. This can be physically represented by simply removing the i^{th} cable in such case or annihilating one degree of robot redundancy to perform non-negative reaction forces in other cables.

It is very important to notice that there exists the normalized nonnegative force $\mathbf{T_i} = \{\tau_{1i} ... \tau_{(i-1)i}\ 0\ \tau_{(i+1)i} ... \tau_{mi}\}$ for each vector of feasible set that the i^{th} component of vector $\mathbf{F}_{feas-i}$ is nonzero. Therefore, there exists a subset of feasible set that the i^{th} component of vector is zero. Also, the minimum solution of (8) is vector of this subset. In order to prove this claim, it is assumed that there exist nonzero minimum value of i^{th} cable denoted by $\mathbf{f}_i^*$, whose component are represented by $f_{1i}^*, ..., f_{mi}^*$. Following with equation (6), the minimum value of $\parallel \mathbf{f} \parallel$ is $\parallel \mathbf{f}_i^* \parallel$. Substitute $\mathbf{f}_i^*$ into (14) and write the resulting equation component-wise:

$$\mathbf{A}_1\tau_{1i}^* + ... + \mathbf{A}_{i-1}\tau_{(i-1)i}^* + \mathbf{A}_{i+1}\tau_{(i+1)i}^* + \mathbf{A}_m\tau_{mi}^* = -\mathbf{w}_i, \tau_{ji}^* = \frac{f_{ji}^*}{(1 + f_{ii}^*)}, \tau_{ii}^* = 0 \tag{16}$$

According to (16), the $(1 + f_{ii}^*)$ is always great than one and the vector of normalized nonnegative force $\mathbf{T_i^*} = \{\tau_{1i}^* ... \tau_{(i-1)i}^*\ \ 0\ \ \tau_{(i+1)i}^* ... \tau_{mi}^*\}$. Compare the value of $\parallel \mathbf{T_i^*} \parallel$ and $\parallel \mathbf{f}_i^* \parallel$, and write the resulting :

$$\parallel \mathbf{T_i^*} \parallel = \parallel \frac{\mathbf{f}_i^*}{(1 + f_{ii}^*)} \parallel \leq \parallel \mathbf{f}_i^* \parallel \tag{17}$$

Therefore, $\mathbf{f}_i^*$ is not the minimum solution and and $\mathbf{T_i^*}$ is the minimum solution. Also, the minimum value of i^{th} cable in $\mathbf{T_i^*}$ is zero. According to given Lemma, fundamental wrench is the worst case wrench that can be generated at the moving platform that one degree of redundancy is annihilated at all configurations. The workspace is obtained from worst wrench belongs to the workspace is obtained from any wrenches. Furthermore, the following theorem may be stated based on fundamental wrench definition and its physical interpretation for a cable parallel robot in order to obtain Controllable workspace.

Theorem: A set of all manipulator configuration ($\mathbf{X}$) belongs to the Controllable workspace, if and only if, there exist a positive semi-definite matrix $\mathbf{T}^* = \{\mathbf{T_1^*} ... \mathbf{T_m^*}\}_{m \times m}$ such that $\mathbf{AT}^* = -\mathbf{W}_F$ for$\mathbf{A}$ is full rank matrix and $\mathbf{W}_F$ is the fundamental wrench matrix .

Proof: *Necessary condition:* If a set of manipulator configuration ($\mathbf{X}$) belongs to the controllable workspace then we must show that there exist a positive semi-definite $\mathbf{T}^* \in \mathbf{R}^{(m \times m)}$ to satisfy $\mathbf{AT}^* = -\mathbf{W}_F$.

Note that if configuration ($\mathbf{X}$) belongs to the controllable workspace, any wrench can be generated at the moving platform while cables are in tension. One of these wrenches is the fundamental wrench. Furthermore, there exist a positive semi-definite $\mathbf{T}^* \in \mathbf{R}^{(m\times m)}$ to satisfy $\mathbf{A}\mathbf{T}^* = -\mathbf{W}_F$.

Sufficient condition: It must be shown that if $\mathbf{T}^* \in \mathbf{R}^{(m\times m)}$ is a positive semi-definite matrix that satisfies $\mathbf{A}\mathbf{T}^* = -\mathbf{W}_F$, the configuration ($\mathbf{X}$) of the manipulator belongs to the controllable workspace.

Let us prove this condition for a general spatial parallel manipulator with one degree of redundancy. For such case, the structure matrix has 6×7 dimension. According to sufficient condition:

$$\exists \mathbf{T}^* \in \mathbf{R}_+^{7\times 7} \ni [\mathbf{A}_1 ... \mathbf{A}_7]_{6\times 7}[\mathbf{T^*}_1 ... \mathbf{T^*}_7]_{7\times 7} = -[\mathbf{w}_1 ... \mathbf{w}_7]_{6\times 7} \tag{18}$$

For each fundamental wrench ($\mathbf{w}_i$), this relation simplifies to:

$$[\mathbf{A}_1 ... \mathbf{A}_7]_{6\times 7}\mathbf{T^*}_i = -\mathbf{w}_i, i = 1...7 \tag{19}$$

Based on Lemma 1, i^{th} element of $\mathbf{T^*}_i$ is zero. For sake of simplicity and without loss of generalization assume $i = 1$. For equation (19), there exist two group of solution. One of them is positive tension force for another cable, i.e. $\tau^*_{1j} > 0,\ \ j = 2...7$.

$$\mathbf{T^*}_1 = [0\ \ \tau^*_{12} ... \tau^*_{17}]^T, \mathbf{w}_1 = \mathbf{A}_1, \tau^*_{1j} > 0, j = 2...7 \tag{20}$$

Substitute equation (20) into equation (19) and write the resulting equation:

$$\mathbf{A}_2\tau^*_{12} + ... + \mathbf{A}_7\tau^*_{17} = -\mathbf{A}_1, \tau^*_{1i} > 0 \tag{21}$$

Move the right hand side component to left hand side:

$$\mathbf{A}_1 + \mathbf{A}_2\tau^*_{12} + ... + \mathbf{A}_7\tau^*_{17} = 0, \tau^*_{1i} > 0 \tag{22}$$

According to equation (22), $[1\ \ \tau^*_{12} ... \tau^*_{17}]^T$ is the positive null space vector of structure matrix $\mathbf{A}$. Furthermore, the configuration (X) of manipulator belongs to the controllable workspace. For another solution, there exist zero value for some of vector i.e., $\tau^*_{1j} \geq 0, j = 2...7$. For sake of simplicity and without loss of generalization assume:

$$\tau^*_{1j} = 0, j = 2...6, \tau^*_{17} > 0 \tag{23}$$

Substitute equation (23) into equation (19) and write the resulting equation:

$$\mathbf{A}_7\tau^*_{17} = -\mathbf{A}_1 \tag{24}$$

Move the right hand side component to left hand side:

$$\mathbf{A}_1 + \mathbf{A}_7\tau^*_{17} = 0 \tag{25}$$

According to equation (25), $[1\ \ 0...0\ \ \tau^*_{17}]^T$ is the nonnegative null space vector of structure matrix $\mathbf{A}$. This configuration does not belong to the controllable workspace. However, this is a solution for equation (19). Based on the theorem, if this configuration belongs to the controllable workspace, the $\mathbf{T}^*$ must be positive semi-definite.

In order to prove, other fundamental wrench is also considered as following. Without loss of generalization assume $i = 2$. According to Lemma (1) and equation (19):

$$\mathbf{A}_1\tau_{21}^* + \mathbf{A}_3\tau_{23}^* + ... + \mathbf{A}_7\tau_{27}^* = -\mathbf{A}_2 \quad (26)$$

Substitute 24 into 26 and write the resulting equation:

$$(-\mathbf{A}_7\tau_{17}^*)\tau_{21}^* + \mathbf{A}_3\tau_{23}^* + ... + \mathbf{A}_7\tau_{27}^* = -\mathbf{A}_2 \quad (27)$$

Move the right hand side component to left hand side:

$$\mathbf{A}_2 + \mathbf{A}_3\tau_{23}^* + ... + \mathbf{A}_7(\tau_{27}^* - \tau_{17}^*\tau_{21}^*) = 0 \quad (28)$$

Noting that $\{\mathbf{A}_2...\mathbf{A}_7\}$ are linear independent, therefore, equation (28) is satisfied if all independent vector coefficient of $\{\mathbf{A}_2...\mathbf{A}_7\}$ are zero. However, in equation (28) the coefficient of $\mathbf{A}_2$ equals to one and the linear independency condition of these vectors cannot be applied. Therefore, similar to previous case, in this situation $\mathbf{T}^*{}_2...\mathbf{T}^*{}_7$ (in the same way) cannot form a positive semi-definite matrix. Therefore, the proof of sufficient condition is completed.

3 Controllable Workspace Analysis

According to the given theorem, the controllable workspace can be determined only by investigation of the existence of solution to $\mathbf{A}\mathbf{T}^* = -\mathbf{W}_F$. One of the distinctive advantages of introducing fundamental wrench is its physical interpretation that it corresponds to zero tension force in each cable, and therefore, this investigation is reduced to looking for the existence of such wrenches in the workspace. An analytical method is described in the following subsection to search for the controllable workspace based on linear algebra approach.

3.1 n DOF with One degree of redundancy

According to the given theorem, the problem is defined as:

$$\mathbf{A}_{n\times(n+1)}\mathbf{T}^* = -\mathbf{W}_F, \mathbf{A} = [\mathbf{A}_1...\mathbf{A}_{n+1}] \quad (29)$$

Based on the Lemma, consider the i^{th} normalized nonnegative force of the cable for i^{th} fundamental wrench is zero. It can be shown that:

$$\mathbf{A}_{n\times(n+1)}[\tau_{i1}^*...\tau_{i(i-1)}^* \;\; 0 \;\; \tau_{i(i+1)}^*...\tau_{i(n+1)}^*]^T = -\mathbf{w}_i, \;\; i = 1...(n+1) \quad (30)$$

According to equation (30), A_i simply remove from equation 30. It can be shown that:

$$[\mathbf{A}_1...\mathbf{A}_{i-1}\mathbf{A}_{i+1}...\mathbf{A}_{n+1}]_{n\times n}[\tau_{i1}^*...\tau_{i(i-1)}^*\tau_{i(i+1)}^*...\tau_{i(n+1)}^*]^T = -\mathbf{w}_i, \;\; i = 1...(n+1) \quad (31)$$

Equation (31) represents n unknown equations with n unknowns, which can be analytically solved as following:

$$\tau_{ij}^* = \frac{\Delta_{ij}}{\Delta_i} = \frac{\det[\mathbf{A}_1...\mathbf{A}_{j-1} \;\; -\mathbf{w}_i \;\; \mathbf{A}_{j+1}...\mathbf{A}_{i-1}\mathbf{A}_{i+1}...\mathbf{A}_{n+1}]}{\det[\mathbf{A}_1...\mathbf{A}_{i-1}\mathbf{A}_{i+1}...\mathbf{A}_{n+1}]} \geq 0 \quad (32)$$

Following with (32), controllable workspace boundary is obtained though following condition:

$$\frac{\Delta_{ij}}{\Delta_i} = 0 \;\; i = 1, ..., n+1 \tag{33}$$

According to (33), it is sufficient to investigate the configurations in which $\Delta_{ij} = 0$. For each fundamental wrench, there exists n equation with n unknown variable. therefore, there exist $n \times (n+1)$ equations of boundary($\Delta_{ij} = 0 \;\; i = 1, ..., n+1$). On the other hand, Δ_{ij}s obtained from i^{th} equation (31) is similar to Δ_{ij}s of another equation. For sake of simplicity and without loss of generalization assume $i = 1, 2$ and $j = 3$ and compare the Δ_{13} and Δ_{23}.

$$\begin{array}{l} \Delta_{13} = \det\left[\mathbf{A}_2 \;\; -\mathbf{w}_1 \;\; \mathbf{A}_4 ... \mathbf{A}_{n+1}\right] = \det\left[\mathbf{A}_2 \;\; -\mathbf{A}_1 \;\; \mathbf{A}_4 ... \mathbf{A}_{n+1}\right] \\ \Delta_{23} = \det\left[\mathbf{A}_1 \;\; -\mathbf{w}_2 \;\; \mathbf{A}_4 ... \mathbf{A}_{n+1}\right] = \det\left[\mathbf{A}_1 \;\; -\mathbf{A}_2 \;\; \mathbf{A}_4 ... \mathbf{A}_{n+1}\right] \end{array} \tag{34}$$

Based on determinant property, Δ_{13} is equal to Δ_{23} in equation (34). Similar to this discussion occur for another boundary equation. Therefore, boundary of controllable workspace is obtained by $(n+1)$ equations of boundary.

3.2 n DOF with more than one degree of redundancy

The controllable workspace analysis problem for n DOF with two degree of redundancy manipulators is defined as following, according to the given theorem:

$$\mathbf{A}_{n\times(n+2)}\mathbf{T}^* = -\mathbf{W}_F, \mathbf{A} = [\mathbf{A}_1 ... \mathbf{A}_{n+2}] \tag{35}$$

For each fundamental wrench the equation (35) is simplified as following:

$$\mathbf{A}_{n\times(n+2)}\mathbf{T}^*_\mathbf{i} = -\mathbf{w}_i \tag{36}$$

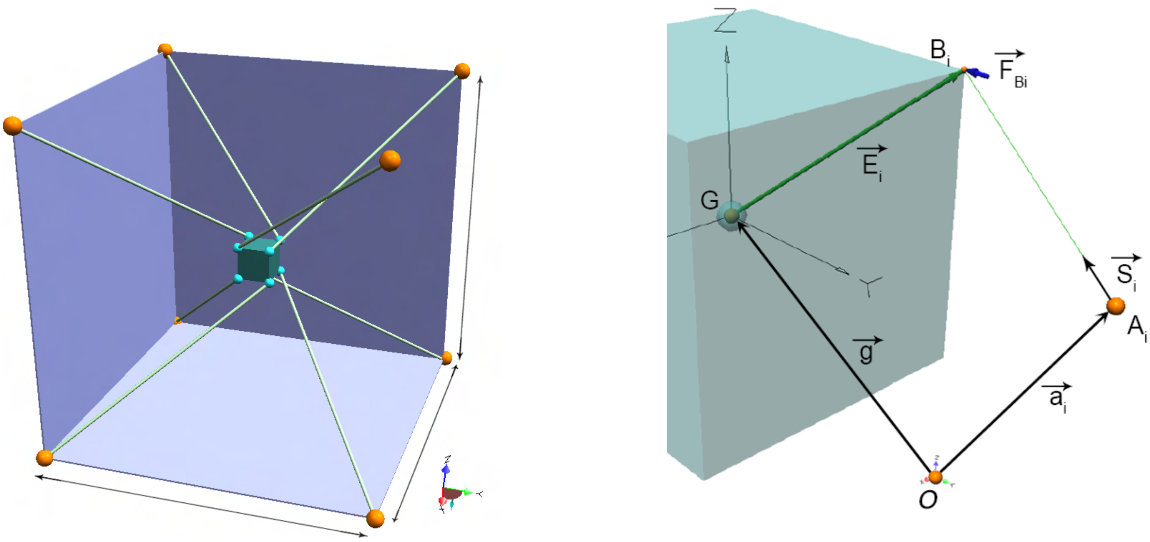
According to lemma I, the i^{th} component of $\mathbf{T}^*_\mathbf{i}$ is zero. therefore, equation (36) is can be simplified as following:

$$[\mathbf{A}_1 ... \mathbf{A}_{i-1}\mathbf{A}_{i+1} ... \mathbf{A}_{n+2}]_{n\times(n+1)}[\tau^*_{i1} ... \tau^*_{i(i-1)}\tau^*_{i(i+1)} ... \tau^*_{i(n+2)}]^T = -\mathbf{w}_i, \;\; i = 1...(n+2) \tag{37}$$

Indeed, right hand side of equation 37 represents right hand side of equation the controllable workspace problem for the n DOF robot with one degree of redundancy. Hence, proposed method in previous section is employed to controllable workspace analysis of this robot. Therefore, the n DOF mechanism with two degree of redundancy is can be distributed to $n+2$ sub-robots with one degree of redundancy. In i^{th} sub-robot, normalized nonnegative force of i^{th} cable of major robot is equal to zero. According to equation (37), the i^{th}sub-robot is the robot without i^{th} cable. $\mathbf{w}_{i+1}$ exerted on end-effector of this sub-robot. Following with Lemma I, equation (37) can be simplified to:

$$[\mathbf{A}_1 ... \mathbf{A}_{i-1}\mathbf{A}_{i+2} ... \mathbf{A}_{n+2}]_{n\times n}[\tau^*_{i1} ... \tau^*_{i(i-1)}\tau^*_{i(i+2)} ... \tau^*_{i(n+2)}]^T = -\mathbf{w}_{i+1} \tag{38}$$

Equation (38) represents n equation with n unknowns . According previous section, the controllable workspace boundray of each sub-robot is obtaned. Finally, the controllable workspace of the major manipulator is obtained by the *union* of controllable workspace of each sub robot.



(a) (b)

Figure 2: (a) The KNTU CDRPM, a perspective view. (b) Vector definitions for Jacobian derivation of KNTU CDRPM.

For investigating the n DOF with r degree of redundancy, the mechanism is distributed to sub-robot with one degree of redundancy. The mechanism with r degree of redundancy is decreased to $n+r$ sub-robots with $n+r-1$ cables. Each of $n+r$ sub-robots is distributed to $n+r-1$ sub-robots with $n+r-2$ cables. So it continues to be distributed to sub-robot with one degree of redundancy. By this means the number of system equation will increase to:

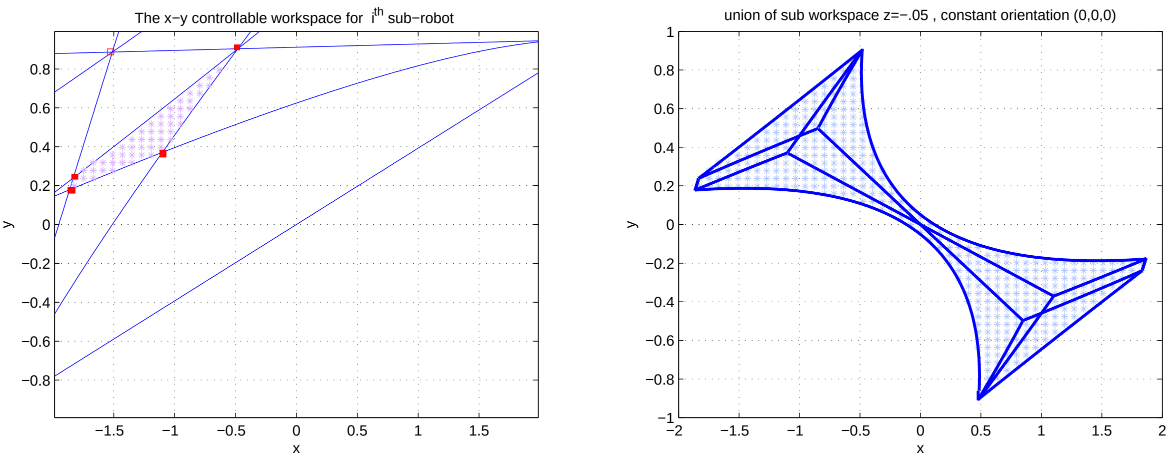
$$C_{r-1}^{n+r} \tag{39}$$

4 Simulation

In this section using the proposed analytic method of determining the controllable workspace, the workspace of a special CDRPM are analyzed. In this section of the controllable workspace of a spatial CDRPM is analyzed. The KNTU CDRPM is designed based on a structure with eight actuators and six degrees of freedom. This manipulator is under investigation for possible high speed and wide workspace applications such as tool manipulation in K. N. Toosi University of Technology [12]. A special design for the KNTU CDRPM is suggested as shown in figure 2(a).

According to obtained result, controllable workspace of manipulator is obtained by union of controllable workspace of each sub robot. Considering the vector definitions $\hat{\mathbf{S}}_i$ and $\mathbf{E}_i$ illustrated in figure 2(b), the manipulator Jacobian matrix $\mathbf{J}$ is derived [12]:

$$\mathbf{J} = \begin{pmatrix} \hat{\mathbf{S}}_1^T & (\mathbf{E}_1 \times \hat{\mathbf{S}}_1)^T \\ \vdots & \vdots \\ \hat{\mathbf{S}}_8^T & (\mathbf{E}_8 \times \hat{\mathbf{S}}_8)^T \end{pmatrix}_{8\times 6} \tag{40}$$



(a) (b)

Figure 3: (a) The curves plotted for i^{th} Fundamental Wrench. (b) Union of sub workspace $([\alpha, \beta, \gamma] = [0, 0, 0], z = -0.05)$

According to the proposed method, the controllable workspace for i^{th} sub-robot due to removing i^{th} is shown in figure 3(a) for $([\alpha, \beta, \gamma] = [0, 0, 0]$ and $z = -0.05)$. In this case, some intersection points of curves lies on the boundary of sub workspace and some of them does not lie on the boundaries. Therefore, active intersection points should be extracted first. An active intersection point is the one lying on the vertex of the boundary of sub workspace. Hence, all intersection points are checked to extract the active ones. For active intersection point of i^{th} sub-robot, the sign of $\frac{\Delta_{ij}}{\Delta_i}$ is non-negative. Also, the workspace is validated by using convex optimization routines such as *fmincon* of MATLAB. The union of controllable workspace of sub robots is shown in figure 3(b) for $([\alpha, \beta, \gamma] = [0, 0, 0]$ and $z = -0.05)$. Some workspace boundaries of sub robots are common to that of other sub robots. Therefore, the number of sub robots workspaces to be analyzed are significantly reduced. Moreover, constant orientation controllable workspace is shown in figure 4 for $([\alpha, \beta, \gamma] = [0, 0, 0])$. It is observed that the controllable workspace of KNTU CDRPM is small region. This might be caused by the structure of the design of this manipulator.

5 CONCLUSIONS

This paper introduces a set of newly defined fundamental wrench for the analysis of controllable workspace of parallel manipulators. Using this definition a physical interpretation of controllable workspace may be given and the complexity of controllable workspace analysis can be significantly reduced. At a given configuration, fundamental wrench is defined as the worst possible exerted wrench on the moving platform, through which one degree of redundancy is annihilated. Through careful examination of the effect of exerting such wrenches on the moving platform, by the given Lemma, and theorem developed in this paper, this physical interpretation is elaborated. Moreover, an analytical method is developed to determine the controllable workspace of redundant cable-driven parallel manipulators based on fundamental wrench. The proposed method is

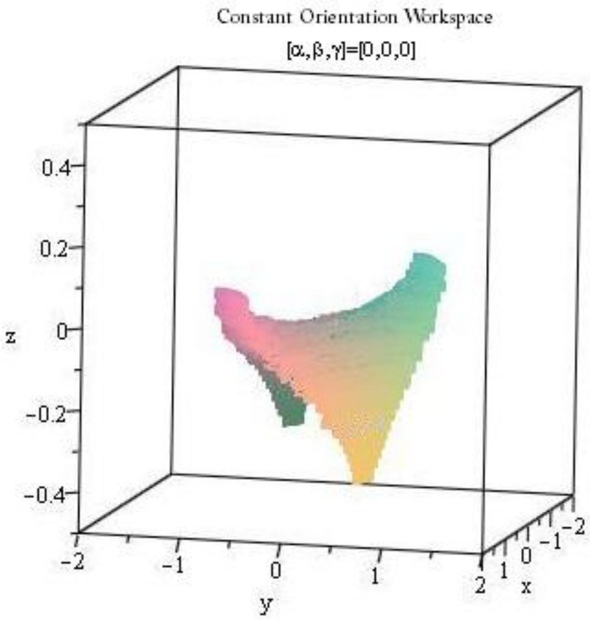


Figure 4: Constant orientation Controllable workspace of KNTU ($[\alpha, \beta, \gamma] = [0, 0, 0]$)

generally applicable to any cable manipulator with any redundant actuation as long as its Jacobian matrix is of full rank. A linear algebra-based approach is employed to determine the boundary of controllable workspace for such manipulators. This method is applied to a spatial manipulator as a case study, and the boundary of controllable workspace is carefully determined for that manipulator. It is believed that this conceptual interpretation of fundamental wrench can be further used as the basis of multi-objective optimization routines including maximization of controllable workspace of cable-driven manipulators.

REFERENCES

[1] Fattah, A., and Agrawal, S. K., On the design of cable-suspended planar parallel robots, Journal of Mechanical Design, Transactions of the ASME, **127**(5):1021-1028, 2005.

[2] Bosscher, P., Riechel, A., Ebert-Uphoff I., Wrench-Feasible Workspace Generation for Cable-Driven Robots, IEEE Transaction on Robotics, **22**(5): 890-902,2006

[3] Ebert-Uphoff, I., and Voglewede, P. A., On the connections between cable-driven robots, parallel robots and grasping, In Proceedings of IEEE InternationalConference on Robotics and Automation, **5**:4521- 4526, 2004.

[4] Barrette, G., and Gosselin, C.M., Determination of the dynamic workspace of cable-driven planar parallel mechanisms, Journal of Mechanical Design, Transactions of the ASME, **127**(2):242-248, 2005.

[5] Verhoeven, R., Hiller, M., Estimating the controllable workspace of tendon-based Stewart platforms, in: Proceedings of the International Symposium on Advances in Robot Kinematics, Portoroz, Slovenia, 277-284, 2002.

[6] Gouttefarde, M., and Gosselin, C. M., Analysis of the wrench-closure workspace of planar parallel cable-driven mechanisms, IEEE Transactions on Robotics, **22**(3):434-445, 2006.

[7] Pham, C.B., Yeo, S.H., Yang, G., Kurbanhusen, M.S., Chen, I-M., force-closure workspace analysis of cable-driven parallel mechanisms, Mechanism and Machine Theory, **41**(1):53-69, 2006.

[8] Fang, S., Franitza, D., Torlo, M., Bekes, F., and Hiller, M., Motion control of a tendon-based parallel manipulator using optimal tension distribution, IEEE/ASME Transactions on Mechatronics, **9**(3):561-568, 2004.

[9] Aref, M. M., Taghirad, H. D., and Barissi, S., Optimal Design of Dexterous Cable Driven Parallel Manipulators, International Journal of Robotics: Theory and Applications, **2**(4): 43-51, 2009.

[10] Nakamura, Y., *Advanced Robotics: Redundancy and Optimization*, Addison-Wesley Longman Publishing Co., 1990.

[11] Taghirad H. D. and Nahon M., Kinematic analysis of a macro-micro redundantly actuated parallel manipulator, Advanced Robotics, **22**(2): 657-687, 2008.

[12] Loloei, A.Z., Aref, M.M., Taghirad, H.D., wrench feasible workspace analysis of Cable-Driven Parallel Manipulators using LMI approach, Advanced Intelligent IEEE/ASME International Conference on Mechatronics AIM09, 1034 - 1039, 2009.