MOBILITY CLASSIFICATION IN THE DESIGN PARAMETER SPACE OF SPHERICAL 4R LINKAGES

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ABSTRACT

The four arc length angles of the links in a spherical 4R mechanism completely determine the mobility of the driver and follower (input and output) links. The design parameter space can therefore be represented by the four arc length angle tangent half-angle parameters α_i, $i \in \{1,2,3,4\}$. Treating these parameters as homogeneous coordinates one can, without affecting the mobility characteristics, project the four-space into the hyperplane of one of the parameters which can be thought of as representing an orthogonal three-space in the remaining parameters. When the three α_i are treated as mutually orthogonal basis directions then the location of a point in the space determines the mobility characteristics of the chain. The algebraic input-output equation of the spherical 4R, an algebraic polynomial in terms of the four α_i and the input and output angle tangent half-angle parameters v_1 and v_4, is a planar quartic curve in v_1 and v_4. Four of the coefficients factor into the product of two cubic surfaces in the four α_i. Each of the eight cubic factors contain linear terms where the four linear α_i possess eight distinct variations in sign. The occurrence or absence of angular displacement limits for the input and output links is completely determined by the signs of products of four of the linear portions of the cubic coefficients, and therefore by the location of a point in the design parameter space spanned by the α_i.

Keywords: Spherical 4R; design parameter space; mobility limits.

CLASSIFICATION DE LA MOBILITÉ DANS L'ESPACE DES PARAMÈTRES DE CONCEPTION DES MÉCANISMES SPHÉRIQUES 4R

RÉSUMÉ

Les quatre angles de longueur d'arc des liaisons dans un mécanisme sphérique 4R déterminent complètement la mobilité des liaisons pilote et suiveur (entrée et sortie). Son espace de paramètres de conception est donc constitué des quatre paramètres de demi-angle tangent d'angle de longueur d'arc α_i, $i \in \{1,2,3,4\}$. En traitant ces paramètres comme des coordonnées homogènes, on peut, sans affecter les caractéristiques de mobilité, projeter les quatre espaces dans l'hyperplan de l'un des paramètres que l'on peut considérer comme représentant un trois espaces orthogonaux dans les paramètres restants. Lorsque les trois α_i sont traités comme des directions de base, l'emplacement d'un point dans l'espace détermine les caractéristiques de mobilité de la chaîne. L'équation algébrique d'entrée-sortie du 4R sphérique, un polynôme algébrique en termes des quatre α_i et des paramètres de demi-angle tangent d'angle d'entrée et de sortie v_1 et v_4, est une courbe quartique planaire dans v_1 et v_4. Quatre des coefficients sont chacun pris en compte dans le produit de deux surfaces cubiques dans les quatre α_i. Chacun des huit facteurs cubiques contient des termes linéaires où les quatre α_i linéaires possèdent huit variations distinctes de signe. L'occurrence ou l'absence de limites de déplacement angulaire pour les liaisons d'entrée et de sortie est complètement déterminée par les signes des produits de quatre des parties linéaires des coefficients cubiques, et donc par l'emplacement d'un point dans l'espace des paramètres de conception enjambé par le α_i.

Mots-clés : Sphérique 4R ; espace des paramètres de conception ; limites de mobilité.

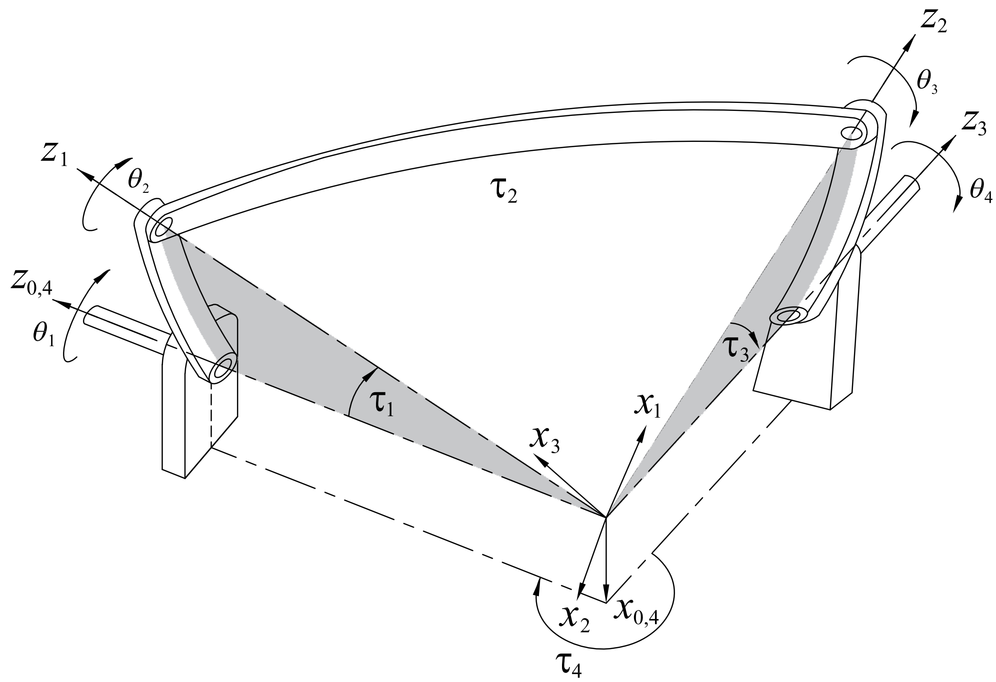


Fig. 1. Spherical 4R DH reference frames and parameters.

1. INTRODUCTION

Spherical 4R linkages have been the focus of research for millennia [1]. Arguably the most successful mechanical system built on spherical 4R closed complex kinematic chains is the Agile Eye [2], introduced in 1994 and used as a camera pointing system. Four years later the mobility conditions on the input and output links in spherical function generators were classified using the trigonometric input-output (IO) equation [3], but 24 years earlier type and mobility considerations were examined [4]. While there is a substantial volume of literature regarding classification, see [5–7] for example, this type of mechanical system still excites the imagination, see [8] for a recent example. Hence, we believe there is sufficient justification to revisit the mobility conditions on the input and output links of spherical 4R mechanisms imposed by the fixed distances between the R-pair centres in light of a novel algebraic IO equation [9].

Consider the arbitrary spherical 4R linkage illustrated in Fig. 1. The IO equation expresses the functional relationship between the input and output angles, $\theta_4 = f(\theta_1)$ in terms of the constant angular distances between the four R-pair centres, τ_i. The derivation of the algebraic form of the spherical IO equation makes use of the original Denavit-Hartenberg (DH) parametrisation of the kinematic geometry [10]. It also requires that all measures of angle be converted to algebraic parameters using the so called *Weierstrass* tangent half-angle substitutions:

$$\begin{aligned} v_i &= \tan\frac{\theta_i}{2}, & \alpha_i &= \tan\frac{\tau_i}{2}; \\ \cos\theta_i &= \frac{1-v_i^2}{1+v_i^2}, & \cos\tau_i &= \frac{1-\alpha_i^2}{1+\alpha_i^2}; \\ \sin\theta_i &= \frac{2v_i}{1+v_i^2}, & \sin\tau_i &= \frac{2\alpha_i}{1+\alpha_i^2}. \end{aligned}$$

In the often bizarre historical record of mathematics facts are sometimes distorted. These half-angle parameters are named after mathematician Karl Weierstrass (1815 - 1897), without any claim of the substitution in Weierstrass' own writings. Indeed, these substitutions are first used in a recognisable way [11] by Leonhard Euler in [12], but come from the much older rational parameterisation of the unit circle which uses the t-line construction and the formulae $x = (1-t^2)/(1+t^2)$, $y = 2t/(1+t^2)$. This substitution goes back in some form to Euclid, or even earlier, who used it to generate Pythagorean triples [13].

Making these substitutions the algebraic form of the IO equation is derived as [9]

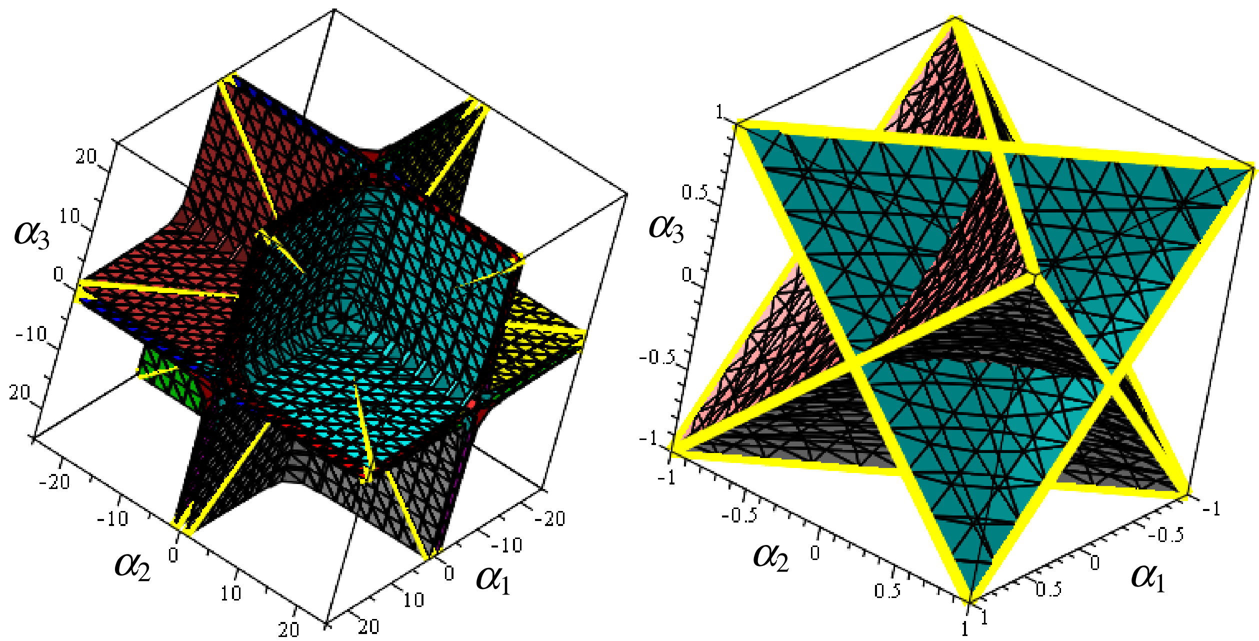
$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 + 8\alpha_1\alpha_3\left(\alpha_4^2+1\right)\left(\alpha_2^2+1\right)v_1v_4 + D = 0, \tag{1}$$

where

$$\begin{aligned}
A = A_1A_2 &= (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)\\
&\quad(\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 - \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4),\\
B = B_1B_2 &= (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)\\
&\quad(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 - \alpha_2\alpha_3\alpha_4 - \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),\\
C = C_1C_2 &= (\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 - \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4)\\
&\quad(\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4),\\
D = D_1D_2 &= (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)\\
&\quad(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 - \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1 - \alpha_2 + \alpha_3 + \alpha_4).
\end{aligned}$$

The eight cubic factors in the four coefficients A, B, C, and D are symmetric singular cubics which each possess three distinct finite lines and three common lines at infinity [14]. Note that a cubic surface can contain as many as 27 lines [15]; those that contain less than 27 are called *singular*, while those that contain exactly 27 are *non-singular*. Different pairs of the eight surfaces have one finite line in common, meaning there are 12 distinct finite lines among the eight surfaces. The finite lines contain the twelve edges of a regular double tetrahedron. The three lines on each surface intersect each other in an equilateral triangle and the eight equilateral triangles form the edges of a stellated octahedron, which has order 48 octahedral symmetry [16]: a regular double tetrahedron that intersects itself in a regular octahedron. Fascinatingly, the faces of the regular double tetrahedron are also found in the design parameter space of planar 4R linkages [14]! The edges of this double tetrahedron can be regarded as the intersection of the linear factors of the coefficients of the planar 4R and the singular cubic surfaces formed by the coefficients of the spherical 4R IO equations in the design parameter spaces. Fig.s 2a and 2b illustrate the eight cubic surfaces and the three finite lines on each.

Without loss in generality, the surfaces are projected into the hyperplane $\alpha_4 = 1$ for visualisation, see Fig. 2. If the α_i are interpreted as directed distances, each distinct point in this space represents a different spherical 4R linkage, while it's location implies the mobility of the input and output links, hence the space is called the design parameter space of spherical 4R linkages. The idea of representing mobility constraints in the space implied by the Freudenstein parameters of planar and spherical 4R linkages was first put forward in [17, 18] in the late 1980's. However the symmetry of the stellated octahedron, which has order 48 octahedral symmetry [16], represented in the design parameter space of planar 4R four-bars which, in a sense, intersects the eight cubic surfaces of the spherical 4R is not present in this representation as the Freudenstein parameter space. Moreover the intersection of the representations of planar and spherical 4R IO equations in the design parameter spaces is not directly observable in that work. We believe that there is something of remarkable beauty in this elegant result: the IO equations in the design parameter spaces of these two classes of mechanism intersect along the edges of the only uniform polyhedral compound in the universe of polyhedra! Regardless, both the Freudenstein approach as with this algebraic approach lead to synthesis equations that are linear in the parameters, which is ideal from a design standpoint. However, the Freudenstein approach is vector-loop based often leading to a single solution, whereas the algebraic approach yields all possible solutions of the synthesis equations.



(a) In the range $-25 \leq \alpha_i \leq 25$.

(b) In the range $-1 \leq \alpha_i \leq 1$.

Fig. 2. Eight cubic surfaces in the spherical 4R design parameter space.

2. MOBILITY CONDITIONS

The magnitudes of the linear components of four of the eight coefficient factors in Eq. (1) determine the mobility of the input and output links leading to results remarkably similar to [19]. Hence, the location of a point in the projection of the design parameter space illustrated in Fig. 2 defines the mobility of a linkage assembled with the links possessing the distances between the R-pairs implied by the values of α_1, α_2, and α_3 with $\alpha_4 = 1$. The effect of normalising α_1, α_2, and α_3 with α_4 is to place the associated function-generator on the surface of a unit sphere, and values of α_4 that are not unity merely scale the angular distances between the R-pairs changing the radius of the sphere but preserving the function correlating the input and output angles. The linear components of interest are contained in the factors A_1, B_1, C_1, and D_1 in Eq. (1), and are correspondingly labelled as

$$\left.\begin{array}{lll} A_{l1} = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4, & & B_{l1} = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4, \\ C_{l1} = -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, & & D_{l1} = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4. \end{array}\right\} \quad (2)$$

The following classification requires that α_4 correspond to the relatively non-moving link.

2.0.1. Existence Condition for $\theta_{1_{\min}}$

Examining the spherical 4R illustrated in Fig. 1 the input link can correspond to either τ_1 or τ_3. We arbitrarily assign the input link to be τ_1 and θ_1 its input angle. If the relative lengths of the links permit, the links corresponding to τ_2 and τ_3 can align on the same great circle. In this configuration θ_1 will be either at its minimum or maximum value. If the arc length of the great circle segment is determined by the angle $\tau_2 - \tau_3$ then θ_1 will be at its minimum value, denoted $\theta_{1_{\min}}$. In order to be able to attain this configuration then it must be that $\cos\theta_{1_{\min}} < 1$. If, on the other hand, $\cos\theta_{1_{\min}} \geq 1$ then the alignment of τ_2 and τ_3 on the same great circle is not mechanically possible and τ_1 will be able to traverse the positive x_0-axis passing through $\theta_1 = 0$. We will consider the condition $\cos\theta_{1_{\min}} = 1$ to be a transition case, meaning that the link lengths in the corresponding mechanism have at least one folding configuration where the input link and the coupler, as well as the output link, lie along the x_0-axis. The condition for this ability to traverse the x_0-axis can be modelled using the law of cosines for spherical triangles [20]

$$\cos\theta_{1_{\min}} = \frac{\cos(\tau_2 - \tau_3) - \cos\tau_1 \cos\tau_4}{\sin\tau_1 \sin\tau_4} \geq 1. \tag{3}$$

Rearranging Eq (3) and using the addition/subtraction identity

$$\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 = \cos(\phi_1 - \phi_2)$$

yields the equivalent condition of

$$\cos(\tau_2 - \tau_3) \geq \cos(\tau_1 - \tau_4). \tag{4}$$

Because the magnitude of the cosine function decreases as the absolute value of its argument increases in the range $0 \leq \Delta\tau \leq \pi$, Eq. (4) can be re-expressed equivalently as

$$(\tau_2 - \tau_3)^2 \leq (\tau_1 - \tau_4)^2, \Rightarrow (\tau_2 - \tau_3)^2 - (\tau_1 - \tau_4)^2 \leq 0. \tag{5}$$

This difference of squares is factored according to $a^2 - b^2 = (a+b)(a-b)$, giving

$$(\tau_1 + \tau_2 - \tau_3 - \tau_4)(-\tau_1 + \tau_2 - \tau_3 + \tau_4) \leq 0. \tag{6}$$

Converting these factors of sums and differences of angles to their algebraic equivalents yields sums and differences of the α_i which correspond to $-A_{l1}$ and B_{l1}, two linear components of the factors listed in Eq. (2), giving

$$\underbrace{(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)}_{B_{l1}}\underbrace{(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)}_{-A_{l1}} \leq 0, \tag{7}$$

or equivalently

$$A_{l1}B_{l1} \geq 0. \tag{8}$$

Hence, the condition for $\theta_{1_{\min}}$ to exist is $A_{l1}B_{l1} < 0$. If, on the other hand, $A_{l1}B_{l1} \geq 0$ then the link defined by α_1 can cross the positive x_0-axis, passing through 0.

2.0.2. Existence Condition for $\theta_{1_{\max}}$

If the relative lengths of the links allow τ_2 and τ_3 to align on the same great circle with arc length determined by $\tau_2 + \tau_3$ then θ_1 will be at its maximum value, denoted $\theta_{1_{\max}}$. The condition enabling link α_1 to pass through π on the x_0-axis, meaning that $\theta_{1_{\max}}$ does not exist, is again modelled with the law of cosines for spherical triangles as:

$$\cos\theta_{1_{\max}} = \frac{\cos(\tau_2 + \tau_3) - \cos\tau_1 \cos\tau_4}{\sin\tau_1 \sin\tau_4} \leq -1. \tag{9}$$

Rearranging Eq. (9) and using the addition/subtraction identity gives the condition

$$\cos(\tau_2+\tau_3) \quad \leq \quad \cos(\tau_1+\tau_4)$$

Now, following the same procedure as for $\theta_{1_{\min}}$ leads to the inequality condition for the non-existence of $\theta_{1_{\max}}$ as the product of the sums and differences of the linear elements listed in Eq. (2)

$$\underbrace{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}_{-D_{l1}}\underbrace{(-\alpha_1+\alpha_2+\alpha_3-\alpha_4)}_{C_{l1}} \quad \geq \quad 0, \tag{10}$$

or

$$C_{l1}D_{l1} \quad \leq \quad 0. \tag{11}$$

In the interest of space, we will only report the mobility classification in the cases where $-D_{l1} > 0$ since the classification is determined in the same way for cases where $-D_{l1} \leq 0$. Therefore, in the case where numerical value for the linear factor D_{l1} is a non-zero negative number, to satisfy the condition in Eq. (11) it must be that

$$C_{l1} \quad \geq \quad 0. \tag{12}$$

Therefore, the condition for $\theta_{1_{\max}}$ to exist is that $C_{l1} < 0$. Alternately, if $C_{l1} \geq 0$ then α_1 can cross the negative x_0-axis, passing through π, in turn meaning that $\theta_{1_{\max}}$ does not exist.

2.0.3. Existence Condition for $\theta_{4_{\min}}$

The procedure for determining the conditions on the link lengths for the existence of a minimum output angle, $\theta_{4_{\min}}$, is similar to that of determining the conditions for θ_1, but uses a different spherical triangle. In order for $\theta_{4_{\min}}$ to exist, then links τ_1 and τ_2 must align on the same great circle with arc length determined by $\tau_2-\tau_1$. If this configuration cannot be reached by the mechanism then $\theta_{4_{\min}}$ does not exist and $\cos\theta_{4_{\min}} \geq 1$, meaning that τ_4, or α_4 depending on how it is represented, can pass through 0 on the x_4-axis. Hence, the condition required for $\theta_{4_{\min}}$ to not exist is given by

$$\cos\theta_{4_{\min}} \quad = \quad \frac{\cos(\tau_2-\tau_1)-\cos\tau_3\cos\tau_4}{\sin\tau_3\sin\tau_4} \;\geq\; 1. \tag{13}$$

The equivalent condition, in the range $0 \leq \Delta\tau \leq \pi$, is given by factoring the difference of squares and converting the τ_i to α_i is

$$\underbrace{(-\alpha_1+\alpha_2+\alpha_3-\alpha_4)}_{C_{l1}}\underbrace{(-\alpha_1+\alpha_2-\alpha_3+\alpha_4)}_{-A_{l1}} \quad \leq \quad 0. \tag{14}$$

This means that the condition for $\theta_{4_{\min}}$ to not exist thus enabling α_3 to pass though 0 is

$$A_{l1}C_{l1} \quad \geq \quad 0. \tag{15}$$

We can conclude that the condition for $\theta_{4_{\min}}$ to exist is that the product of linear factors $A_{l1}C_{l1} < 0$. Alternately, if $A_{l1}C_{l1} \geq 0$ then α_4 can cross the positive x_4-axis, passing through 0, in turn meaning that $\theta_{4_{\min}}$ does not exist.

2.0.4. Existence Condition for $\theta_{4_{\max}}$

The condition for the existence of $\theta_{4_{\max}}$ is that links τ_1 and τ_2 must align on the same great circle with arc length determined by $\tau_1+\tau_2$. For α_4 to have the ability to pass through π on the x_4-axis is given by $\cos\theta_{4_{\max}} \leq -1$, meaning that

$$\cos\theta_{4_{\max}} = \frac{\cos(\tau_1+\tau_2)-\cos\tau_3\cos\tau_4}{\sin\tau_3\sin\tau_4} \leq -1. \tag{16}$$

Following the same procedure detailed as for $\theta_{1_{\max}}$ leads to the condition for the non-existence of $\theta_{4_{\max}}$ as

$$\underbrace{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}_{-D_{l1}}\underbrace{(\alpha_1+\alpha_2-\alpha_3-\alpha_4)}_{B_{l1}} \geq 0. \tag{17}$$

This means that the condition for $\theta_{4_{\max}}$ to not exist is

$$B_{l1}D_{l1} \leq 0. \tag{18}$$

As with the conditions on the existence of $\theta_{1_{\max}}$ we may consider only the cases where $-D_{l1}>0$ and consider only the linear factor B_{l1} in the non-existence criterion in Eq. (18) and restate the condition for $\theta_{4_{\max}}$ to not exist thereby enabling α_3 to pass though π is

$$B_{l1} \geq 0. \tag{19}$$

The existence criteria for the minimum and maximum input and output joint angles in every spherical 4R linkage where $D_{l1}<0$ are summarised in Table 1.

Angle	Exists if	Does not exist if
$\theta_{1_{\min}}$	$A_{l1}B_{l1}<0$	$A_{l1}B_{l1}\geq 0$
$\theta_{1_{\max}}$	$C_{l1}<0$	$C_{l1}\geq 0$
$\theta_{4_{\min}}$	$A_{l1}C_{l1}<0$	$A_{l1}C_{l1}\geq 0$
$\theta_{4_{\max}}$	$B_{l1}<0$	$B_{l1}\geq 0$

Table 1. Existence criteria for minimum and maximum input and output joint angles in spherical 4Rs where $D_{l1}<0$.

2.1. Mobility Classification for Spherical 4R Linkages

It is to be seen that the magnitude of four of the linear components, A_{l1}, B_{l1}, C_{l1}, and D_{l1} of the eight cubic factors of the coefficients of Eq. (1) completely determines the mobility of the input and output links. If we classify the mobility of the linkages limiting our possibilities to allowing A_{l1}, B_{l1}, C_{l1} to have any one of the three values $(-,\ 0,\ +)$ while $-D_{l1}>0$, then there are $3^3=27$ permutations. If $-D_{l1}$ is also allowed to vary in value in the same way then there will be $3^4=81$, or 54 additional linkage mobility classes, all classified in the same way. The eight distinct mobility types in the 27 cases where $-D_{l1}>0$ are listed in Table 2. Depending on the twist angle parameters and sphere radius, each of the first three of the four linear components can be positive, negative, or identically zero, while D_{l1} is always less than zero. In the classification scheme first presented in [3] and later refined in [21] trigonometric relations are only

#	A_{l1}	B_{l1}	C_{l1}	Input α_1	Output α_4	#	A_{l1}	B_{l1}	C_{l1}	Input α_1	Output α_4
1	+	+	+	crank	crank	15	0	0	-	0-rocker	crank
2	+	+	0	crank	crank	16	0	-	+	crank	0-rocker
3	+	+	-	0-rocker	π-rocker	17	0	-	0	crank	0-rocker
4	+	0	+	crank	crank	18	0	-	-	0-rocker	0-rocker
5	+	0	0	crank	crank	19	-	+	+	π-rocker	π-rocker
6	+	0	-	0-rocker	π-rocker	20	-	+	0	π-rocker	crank
7	+	-	+	π-rocker	0-rocker	21	-	+	-	rocker	crank
8	+	-	0	π-rocker	0-rocker	22	-	0	+	crank	π-rocker
9	+	-	-	rocker	rocker	23	-	0	0	crank	crank
10	0	+	+	crank	crank	24	-	0	-	0-rocker	crank
11	0	+	0	crank	crank	25	-	-	+	crank	rocker
12	0	+	-	0-rocker	crank	26	-	-	0	crank	0-rocker
13	0	0	+	crank	crank	27	-	-	-	0-rocker	0-rocker
14	0	0	0	crank	crank						

Table 2. Classification of all possible planar spherical 4R linkages where $D_{l1} < 0$. Shaded cells satisfy the Grashof condition.

considered. Because the sum of any two angles in a spherical triangle can exceed π, but not 2π, while the sum of the three interior angles is greater than π, but less than 3π, it may happen that the argument of the cosine function is not in the range between 0 and π. To address this the trigonometric form of the D_{l1} term is modified to

$$D'_{l1} = 2\pi - \tau_1 - \tau_2 - \tau_3 - \tau_4.$$

Depending on the magnitudes of the angles D'_{l1} may be less than, greater than, or identically equal to 0. If $D'_{l1} < 0$ then the linkage wraps around the sphere [21]. Regardless, for each of the eight possible mechanism types possessing mobility determined by the signs of the other three linear factors is precisely the same as those for the case where $D'_{l1} > 0$. Moreover, when converted to their algebraic parameters we see that $D'_{l1} = D_{l1}$ since $\tan 2\pi/2 = 0$. Therefore we only consider the eight cases where $D_{l1} < 0$ which aligns with results already established in the literature [3, 21].

Moreover the Grashof condition is satisfied when the product $A_{l1}B_{l1}C_{l1}D_{l1} < 0$. The four possible Grashof linkages are the shaded cells in Table 2. If the link lengths permit any one, or any combination of A_{l1}, B_{l1}, or C_{l1} to be identically zero, then the linkage can fold, these additional 19 folding linkages are also tabulated. The number of linear factors equalling zero corresponds to the number of ways the linkage can fold.

3. CONCLUSIONS

In this paper, using the algebraic IO equation for spherical 4R linkages from [9], we have shown that the linear elements of four of the eight cubic factors of link lengths play a role in characterising the mobility characteristics of the input and output links. Any point in the design parameter space of the link length parameters α_1, α_2, and α_3 projected into the hyperplane $\alpha_4 = 1$ establishes the mobility characteristics listed in Table 2. While the results themselves are not new, the method by which they are obtained is. Moreover, since the same approach can be applied to planar 4R linkages as well, we are steadily approaching the development of a completely general function generator algebraic IO equation derivation algorithm for any planar, spherical, or spatial four-bar linkage kinematic architecture which reveals the associated design parameter spaces.

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