The Three-Points-on-Three-Lines Problem

Paul Zsombor-Murray *Jonathan Shum*
Dept. Mech. Eng. & Center for Intelligent Machines
McGill University, Montréal, Canada

Abstract: The three-points-on-three-lines problem, a direct kinematic analysis paradigm for the spatial double triangular parallel manipulators(R$\Delta\Delta$PM), can be solved by intersecting three mutually orthogonal cylinders in the parameter space. The minimum solution is found to be of order 8, however only 2 of the 8 resulting poses lie within the workspace.

1. Introduction

The novel *double triangular*($\Delta\Delta$) architecture was proposed and studied in depth by Daniali [1],...,[3] to provide for applications which require fast, precise motion. The conceptual model of this novelty, as shown in Fig. 1, consists of a pair of general or degenerate triangles placed one on top of the other. The upper is moved by sliding the edge pair intersection points along the three fixed edges. This arrangement provides legs of essentially zero length and hence allows quick and accurate positioning without problems associated with long, slender legs.

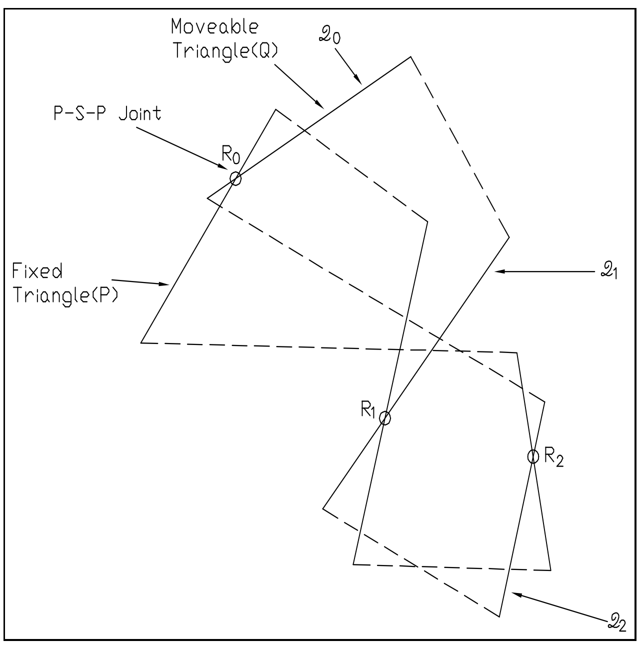

Figure 1. Conceptual Model of R$\Delta\Delta$PM

The direct kinematic problem, as shown in Fig. 2, is solved by placing three points (the joint inputs $R_i, i = 1,\ldots,3$) on the corresponding edges of the moveable triangle ($\mathcal{Q}_i, i = 1,\ldots,3$), which assumes the rôle of temporary fixed frame. This is the so called *three-points-on-three-lines* problem. The final pose of the moveable triangle can then be determined by reverse displacement of the joints back to its original position on the fixed triangle.

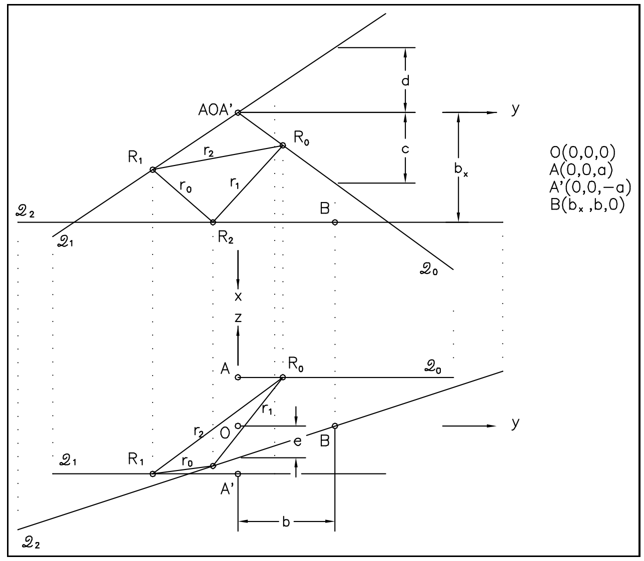

Figure 2. Kinematic Model of R$\Delta\Delta$PM

2. General Solution

The coordinate system for this problem is chosen so that lines $\mathcal{Q}_0$ and $\mathcal{Q}_1$ intersect the z-axis at $z = \pm a$, respectively. Furthermore $\mathcal{Q}_2$ is placed on the plane $x = b_x$. All this can be done without violating rigid body properties of the frame which contains these lines. The equations of the three edges of the moveable *spatial* triangle Q with skew edges $\mathcal{Q}_0, \mathcal{Q}_1$ and $\mathcal{Q}_2$, can be written parametrically by using scalar variables u, v and w

$$\mathcal{Q}_0 : \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} + u \begin{bmatrix} c \\ b \\ 0 \end{bmatrix}$$

$$\mathcal{Q}_1 : \begin{bmatrix} 0 \\ 0 \\ -a \end{bmatrix} + v \begin{bmatrix} d \\ b \\ 0 \end{bmatrix}$$

$$\mathcal{Q}_2 : \begin{bmatrix} b_x \\ b \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ b \\ e \end{bmatrix}$$

The problem is solved by noting that the joint triangle vertices (R_0, R_1 and R_2) must be placed on respective sides of Q, with their respective opposite sides of known length r_0, r_1 and r_2. Three sphere equations can be written as follows:

The first sphere, radius r_2 centered on $\mathcal{Q}_0$, intersects $\mathcal{Q}_1$ at two points $\mathcal{Q}_{1a}$ and $\mathcal{Q}_{1b}$. The equation can be written as

$$||(\mathcal{Q}_1 - \mathcal{Q}_0)||^2 - r_2^2 = 0 \tag{1}$$

Step 2 requires the construction of "second spheres" with radius r_0 centered on points $\mathcal{Q}_{1i}, i = 1, 2$, which intersects $\mathcal{Q}_2$ at four points $\mathcal{Q}_{2ia}$ and $\mathcal{Q}_{2ib}$. The equation can be written as

$$||(\mathcal{Q}_2 - \mathcal{Q}_1)||^2 - r_0^2 = 0 \quad (2)$$

Finally, the "third sphere", radius r_1 centered on $\mathcal{Q}_0$, must also intersect $\mathcal{Q}_2$ at the points $\mathcal{Q}_{2ij}, i, j = 1, 2$ in order to form a compatible joint triangle. The equation can be written as

$$||(\mathcal{Q}_2 - \mathcal{Q}_0)||^2 - r_1^2 = 0 \quad (3)$$

Since Eqs. (1), (2) and (3) are quadratic in (u,v), (v,w) and (u,w) respectively, geometrically the three equations are three mutually orthogonal cylinders of conic section in the parameter space (u,v,w). The solutions of this system of equations are the intersections of the three cylinders. A univariate in u can be obtained by first solving for $v = v(u)$ and $w = w(u)$ using Eqs. (1) and (3), then back substitute into Eq. (2) will produce an eighth order univartate in u.

3. Special Case: The Conical ΔΔPM

Aside from the planar ΔΔPM, which has a quadratic univariate forward kinematic solution, there is at least one other special variant of 3dof ΔΔPM that has a forward kinematic univariate of less than eight. In this case all three straight lines of the moveable triangle intersect at a common point. One sees architecture of this type in at least one existing manipulator design [6].

A Problem in Statics

Many years ago Jack Philips posed the problem concerning how a rigid body, supported on three strings, attached to three given points on the body and anchored at three given points, would hang [4],[5]. Notice that the solution to this problem is identical to that of the conical ΔΔPM forward kinematics.

Formulation

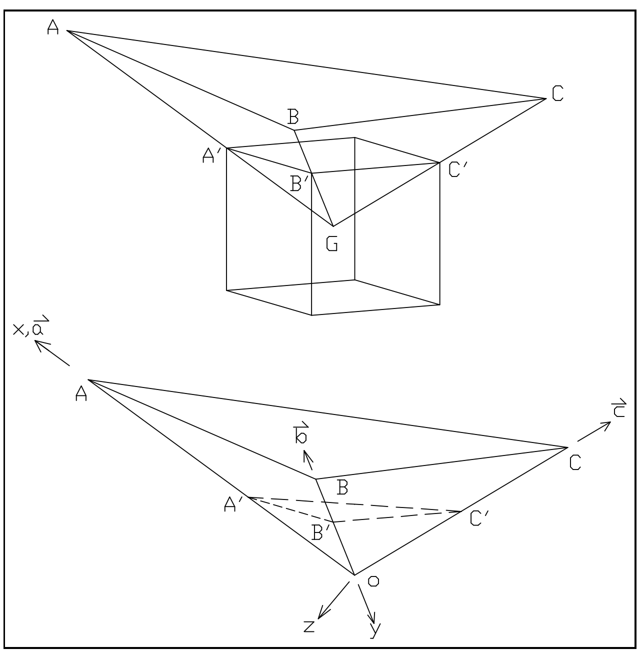

Figure 3. The Hanging Box Problem

Upon examining Fig. 3 one sees the hanging box-three point hitch configuration where the applied force is concentrated to act through G, while being sustained by three components acting along $GA'A, GB'B, GC'C$. The string-tiebars span $A'A, B'B, C'C$. Whether dealing with the conical ΔΔPM or the hanging box, the problem is set up in the simplifying plane shown by applying the tetrahedron $OA'B'C'$ such that O is on the origin, A' is along the x-axis and B' is on the plane z=0. Parametrization is similar to that adopted for the general spatial ΔΔPM. The position vectors $\vec{a}, \vec{b}, \vec{c}$ of points A, B, C can be written in terms of the origin-to-point parameters u, v, w.

$$\vec{a} = u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b} = v \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}, \quad \vec{c} = w \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Using the method as described for the general case, will yield an eighth order univariate in u that contains only even powers. Therefore the conical ΔΔPM DK problem is a fourth order univariate in u^2. Numerical trials show that only 2 of the 8 solutions lie within the workspace (tetrahedron $OABC$).

4. Conclusion

- The *three-points-on-three-lines* formulation can be used to solve the direct kinematic problem of spatial parallel manipulators as well as problem in statics.
- In general, the minimum solution for the problem is a univariate of order eight. In the special case when the three edges intersects at one point the eighth order univariate contains only even powers.
- Only 2 of the eight solutions lie within the workspace. This observation provides insights in the analysis of the spherical ΔΔPM.

References

[1] Daniali, H.R.M., Zsombor-Murray, P.J. & Angeles, J., (1996), Direct Kinematics of Double-Triangular Parallel Manipulators, *Mathematica Pannonica*, v.7, n.1, 1996, pp.79-96.

[2] Daniali, H.R.M., (1995), Contributions to the Kinematic Synthesis of Parallel Manipulators, Ph.D. thesis (Dean's Honours List), McGill University, 95-04.

[3] Daniali, H.R.M., Zsombor-Murray, P.J. & Angeles, J., (1995), The Kinematics of Spatial Double-Triangular Manipulators, *ASME J. Mechanical Design*, v.117, n.4, 1995, pp.658-661.

[4] Philips, J., *Freedom in Machinary, vol. 1, Introducing Screw Theory*, Cambridge Univ. Press, 1984.

[5] Philips, J., *Freedom in Machinary, vol. 2, Screw Theory Examplified*, Cambridge Univ. Press, 1990.

[6] Siciliano, B., "A Study on the Kinematics of a Class of Parallel Manipulators", *Advances in Robot Kinematics: Analysis and Control*, Lenarčič, J. & Husty, M. (eds.), Kluwer, 1998, pp.29-38.