LAGRANGIAN DYNAMICS OF CABLE-DRIVEN PARALLEL MANIPULATORS:

A VARIABLE MASS FORMULATION

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Abstract

In this paper the dynamic analysis of cable-driven parallel manipulators (CDPMs) is performed using Lagrange formulation. The effect of entering mass stream to the system caused by elongation of the cables is treated by using Lagrange variable mass formulation. By these means, a complete dynamic model of the system is derived, while the compact and tractable closed-form dynamics formulation is preserved. In this treatment, first a general formulation for a general CDPM is given, while the effect of change of mass in the cables is integrated into its dynamics. The significance of such treatment is appreciated in a complete analysis of the dynamics, vibrations, stability of such systems, and in any robust control synthesis of such manipulators. The formulations obtained for such system is applied to a typical planar CDPM. Through numerical simulations the validity and integrity of the obtained formulations are firstly verified, and then significance of variable mass treatment in such analysis is examined. For this example, it is shown that the effect of entering mass stream into the system is not negligible, while it is non-linear and strongly dependent to the geometric and inertial parameters of the robot, as well as the maneuvering trajectory.

Keywords: cable-driven parallel manipulator (CDPM), variable mass Lagrange formulation, closed-form dynamics.

APPLICATION DE LA MÉTHODE DE LAGRANGE POUR LA MODÉLISATION DES ROBOTS À CABLES : UNE FORMULATION À MASSES VARIABLES

Résumé

Dans cet article, la modélisation dynamique des manipulateurs parallèles à câbles est présentée. L'effet de la variation de la longueur des câbles est pris en compte grâce à la méthode de Lagrange pour des systèmes à masses variables. Le modèle dynamique obtenu se présente alors sous sa forme compacte. Cette modélisation est importante pour étudier en détails la dynamique du système aussi bien que pour des études de vibration, de stabilité et de conception de système de commande robuste. Un exemple de modélisation d'un manipulateur à câble de type planaire est également présenté en détails. Grâce à des simulations numériques, la validité et l'intégrité de la formulation obtenue sont d'abord vérifiés. Ensuite, l'effet de la variation de la masse est examiné. Pour ce faire, des simulations avec et sans l'effet de la variation de la masse sont considérées et les résultats sont comparés. Il est montré que pour l'exemple présenté, l'effet de la variation de la masse ne peut pas être négligé. Cette effet est non-linéaire et dépend fortement de la géométrie du manipulateur ainsi que de la trajectoire du robot.

Mots-clés : Robots parallèles à câbles; Lagrange pour les masses variables; Dynamique de chaînes mécaniques fermées.

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1 INTRODOUCTION

The equations of motion of the constant mass systems can be derived using different classical approaches such as Newton-Euler, Lagrange, virtual work, or Kane's formulations. These basic principles of classical dynamics are usually treated for systems comprising definite number of objects with constant masses [1]. These methods can be extended for cases, in which the masses of the system components are changing. Such complete treatment of dynamic analysis of systems with variable mass is a challenging problem. The difficulties arise from the fact that in such mechanisms, the mass, the center of mass and the moments of inertia may vary by overtaking or expelling a stream mass at a non-zero velocity. Consequently the mass that is overtaken or expelled from the system may change the linear and angular momentum of the overall system [2]. The dynamics of variable-mass systems have been studied for a very long time. In the applied mechanics “Continuously mass variable systems”, such as rockets were among the first applications of variable mass systems [3], and usually the first works reported in this area are mostly related to these applications. Meshchersky was among the first scientists that realized the foundation of modern dynamics of a rigid body with variable mass [4]. On the other hand, in robotics applications, robots that pick up lifting objects may be treated by varying mass dynamics. Representatives of such analysis may be named as the work that has been done by McPhee (1991) in dynamics analysis of multi-rigid-body variable mass systems [5]. Furthermore, Djerassi (1998) [6], reported similar works in such robotics applications. The most recent works reported in the area of variable mass system is performed by Cveticanin [2, 4, 7-10]. She carefully studies the dynamics of body separation and developed an analytical procedure to determine the dynamic parameters of the remaining body after mass separation [10]. This method is based on the general principles of momentum and angular momentum of a system of bodies. She has also extended the Lagrange formulation for the systems of varying mass [2]. In the latest reported work of Cveticanin and Djukic, extended kinematic and dynamic properties of a body in general motion is elaborated [9] and the principle of linear and angular momentum conservation were modified to obtain the linear and angular velocity of the body during mass separation.
Furthermore, the dynamics analysis of cable-driven parallel manipulators (CDPMs) shows inherent complexity due to their closed-loop structure and kinematic constraints. Although the dynamic analysis of such manipulators is essential for stability analysis and closed-loop control synthesis, there are few

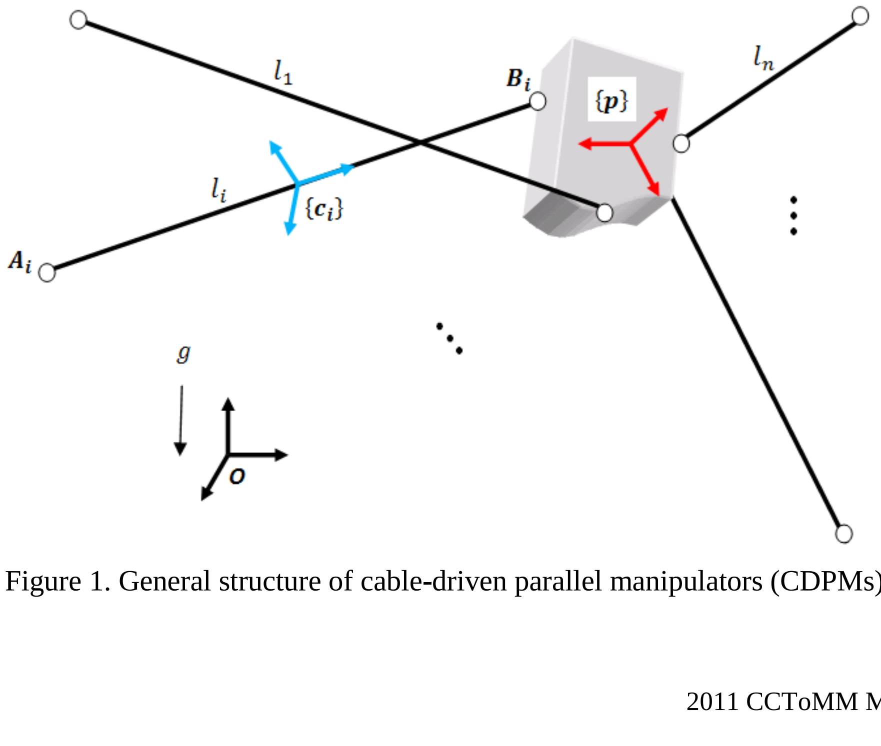


Figure 1. General structure of cable-driven parallel manipulators (CDPMs).

works reported in the dynamic analysis of cable driven parallel manipulators [11-14]. In CDPMs, the change of cable length will cause the effective mass of limbs to be continuously varying in time. Moreover, the varying mass in cables is a function of the position of the moving-platform. In all works that have been reported in the dynamics of CDPMs, the effect of varying mass in cables is neglected due to the small changes of mass in the cables. However, in some applications such as in large adaptive reflectors used in the next generation of giant telescopes [12], the length of cables can be as long as 1000 meters, and the mass variation of cables plays a vital role in the dynamics of the manipulator.
In this paper, the dynamic equations of CDPMs will be discussed in detail by Lagrange formulation and a set of compact and closed-form formulations are obtained. Furthermore, the effect of varying mass in cables is carefully analyzed in the dynamics of the manipulator. Finally, this general formulation is adopted for a typical planar CDPM, for which a simulation study is performed. It is shown that in that case the effect of entering mass stream into the system is not negligible, while it is non-linear and strongly dependent to the geometric and inertial parameters of the robot, as well as the maneuvering trajectory.

2 KINEMATICS ANALYSIS OF CDPM

The general structure of CDPMs that is used in this paper is shown in figure 1. In this manipulator the moving platform is supported by n limbs (cables) of identical kinematic structure, while the limbs are considered as rigid slender rods for the sake of dynamic analysis. The kinematic structure of the limb may be considered as spherical-prismatic-spherical, in which only the prismatic joint is actuated (commonly denoted as SPS). The kinematic structure of a prismatic joint is used to model the elongation of each link. As it is shown in figure 2, A_i denote the fixed base points of the cables, B_i denote the attachment point of the cables to the moving platform, and $\mathbf{l} = [l_1 \quad \dots \quad l_n]$ denote the vector of cable lengths. Moreover, the position vector of the moving platform frame $\{p\}$ as well as the cable frame $\{c_i\}$ are defined as $\left[\mathbf{x}_p^T \quad \mathbf{x}_c^T\right]^T$, in which, $\mathbf{x}_p$ denotes the position of moving platform according to the base frame $\{0\}$ and $\mathbf{x}_c = [\mathbf{x}_{c1}^T \quad \dots \quad \mathbf{x}_{cn}^T]^T$ denotes the vector of the cable coordinates where $\mathbf{x}_{ci}^T$ is the position of the cable center c_i according to the base frame (see figure 2). Similarly, the angular coordinate of the moving-platform $\{p\}$ and the cables $\{c_i\}$ relative to the base frame are defined as $\left[\boldsymbol{\varphi}_p^T \quad \boldsymbol{\varphi}_c^T\right]^T$, in which, $\boldsymbol{\varphi}_p = [\gamma \quad \beta \quad \alpha]^T$are any user-defined Euler angles of the moving platform and $\boldsymbol{\varphi}_c = [\boldsymbol{\varphi}_1^T \quad \dots \quad \boldsymbol{\varphi}_n^T]^T$is the angle vectors of the coordinates attached to the center of the

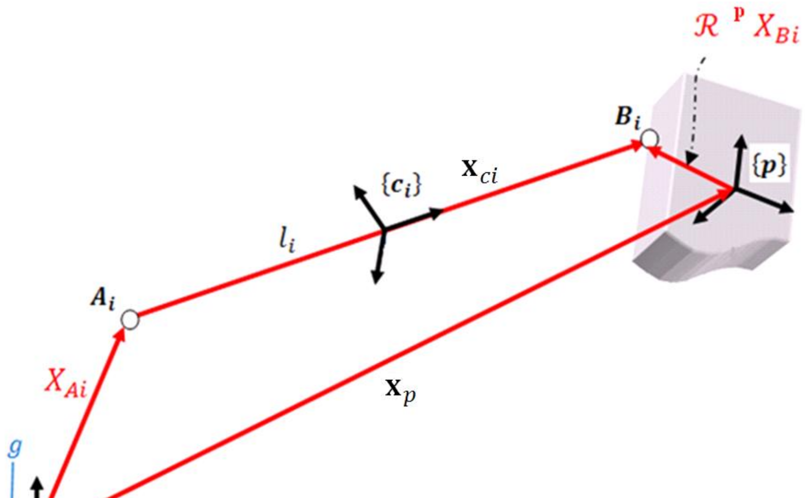


Figure 2. A single limb in a cable-driven parallel manipulator.

cables. Subsequently, each angle vector is defined by its three Euler angles $\boldsymbol{\varphi}_i = [\gamma_i \quad \beta_i \quad \alpha_i]^T$. Accordantly, we consider the following rotation matrices:

$$\begin{aligned} \mathbf{R}(\gamma,\beta,\alpha) &= {}^{O}\mathbf{R}_p, \\ \mathbf{R}(\gamma_i,\beta_i,\alpha_i) &= {}^{O}\mathbf{R}_{c_i}. \end{aligned} \tag{1}$$

As explained in [12], [15], and [16], inverse kinematics of CDPMs, like any other parallel manipulator, can be obtained by writing the loop-closure equations. These equations allow all coordinates of the system to be expressed as function of the generalized coordinates. By choosing $\mathbf{x} = [\mathbf{x}_p^T \quad \boldsymbol{\varphi}_p^T]^T \in \mathbb{R}^m$ (moving-platform position and orientation) as generalized coordinates, one obtains:

$$\mathbf{x}_c = \mathbf{f}_x(\mathbf{x}), \ \ \boldsymbol{\varphi}_c = \mathbf{f}_{\boldsymbol{\varphi}}(\mathbf{x}), \ \ \mathbf{l} = \mathbf{f}_l(\mathbf{x}), \tag{2}$$

where $\mathbf{f}_x$, $\mathbf{f}_{\boldsymbol{\varphi}}$and $\mathbf{f}_l$ are kinematic equations obtained from the loop-closure. Time derivative of (2) may lead to a relation that expresses the linear and angular velocities of the cables as well as the time derivative of the cable lengths as function of the linear and angular velocities of the moving platform:

$$\begin{bmatrix} \dot{\mathbf{x}}_c \\ \boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x}) \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \boldsymbol{\omega}_p \end{bmatrix}, \ \dot{\mathbf{l}} = [\mathbf{J}_{lx}(\mathbf{x}) \quad \mathbf{J}_{l\omega}(\mathbf{x})] \begin{bmatrix} \dot{\mathbf{x}}_p \\ \boldsymbol{\omega}_p \end{bmatrix}, \tag{3}$$

where $\mathbf{J}_{xx}$, $\mathbf{J}_{x\omega}$, $\mathbf{J}_{\omega x}$ and $\mathbf{J}_{\omega\omega}$ are Jacobian matrices $\dot{\mathbf{x}}_c$ and $\dot{\mathbf{x}}_p$ are the linear velocities of the cables and the moving-platform respectively, and $\boldsymbol{\omega}_c$ and $\boldsymbol{\omega}_p$ are the angular velocities expressed in the cables and moving-platform frame respectively. In order to eliminate the velocities of the cable into the Lagrange formulation presented below, (3) is used by collecting all linear velocities of the cables and the moving-platform as function of only the linear and angular velocities of the moving-platform:

$$\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \boldsymbol{\omega}_p \end{bmatrix}. \tag{4}$$

Similarly, the angular velocities of the cables and the moving-platform are rewritten as:

$$\begin{bmatrix} \boldsymbol{\omega}_p \\ \boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \boldsymbol{\omega}_p \end{bmatrix}. \tag{5}$$

By convenience to the Lagrange formulation, (4) and (5) can be expressed as function of the derivative of the generalized coordinates. In order to do that, the following relation between the derivative of the Euler angles and the angular velocity can be established [15]:

$$\boldsymbol{\omega}_p = \mathbf{J}_{\omega\varphi}(\mathbf{x})\dot{\boldsymbol{\varphi}}_p. \tag{6}$$

This equation can then be used to rewrite (4) and (5) as:

$$\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\boldsymbol{\varphi}}_p \end{bmatrix} = \mathbf{J}_x(\mathbf{x})\dot{\mathbf{x}}, \tag{7}$$

$$\begin{bmatrix} \boldsymbol{\omega}_p \\ \boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{J}_{\omega\varphi}(\mathbf{x}) \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\boldsymbol{\varphi}}_p \end{bmatrix} = \mathbf{J}_{\boldsymbol{\varphi}}(\mathbf{x})\dot{\mathbf{x}}. \tag{8}$$

3 KINETIC ENERGY OF CDPM

In order to derive the kinetic energy of the system, the kinetic energy of the robot components are derived and added. A CDPM consists of a moving-platform and several limbs, in which the limbs are modeled as rigid slender rods. Therefore the mass of all objects in the mechanism can be expressed as:

$$\mathbf{M}(\mathbf{l}) = \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_c(\mathbf{l}) \end{bmatrix}, \tag{9}$$

in which, $\mathbf{M}_p$and $\mathbf{M}_c$ denote the mass matrices of the moving platform and all the cables, respectively:

$$\mathbf{M}_p = \begin{bmatrix} m_p & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & m_p \end{bmatrix}, \quad \mathbf{M}_c(\mathbf{l}) = \begin{bmatrix} m_{c1}\mathbf{I}_3(l_1) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & m_{cn}\mathbf{I}_3(l_n) \end{bmatrix}. \tag{10}$$

In this definition, m_p is the moving-platform mass and m_{ci} is the mass of the cables expressed as function of its density ρ_m and its lengths l_i as follows:

$$m_i(l_i) = \rho_m l_i. \tag{11}$$

Similarly, the moment of inertia of all components of CDPM can be collected into:

$$\mathbf{I}(\mathbf{l}) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_c(\mathbf{l}) \end{bmatrix}, \tag{12}$$

where $\mathbf{I}_p$ and $\mathbf{I}_c$ are the inertia matrices of the moving-platform and the cables, respectively given by:

$$\mathbf{I}_p = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}, \ \mathbf{I}_c(\mathbf{l}) = \begin{bmatrix} \mathbf{I}_{c1}(l_1) & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{cn}(l_n) \end{bmatrix}. \tag{13}$$

Since the cables are modeled as slender rods, the moment of inertia of the cables $\mathbf{I}_{ci}$ is defined as:

$$\mathbf{I}_{ci}(l_i) = \frac{\rho_m}{12} \begin{bmatrix} l_i^{\ 3} & 0 & 0 \\ 0 & l_i^{\ 3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

According to (2), l_i can be expressed as function of the generalized coordinates. Thus, the total kinetic energy for all components of a CDPM can be expressed as:

$$T = \frac{1}{2}\left[\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix}^{\boldsymbol{T}} \mathbf{M}(\mathbf{x}) \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}_p \\ \boldsymbol{\omega}_c \end{bmatrix}^T \mathbf{I}(\mathbf{x}) \begin{bmatrix} \boldsymbol{\omega}_p \\ \boldsymbol{\omega}_c \end{bmatrix} \right]. \tag{15}$$

The substitution of the Jacobian matrices defined by Equations (7) and (8) leads to:

$$T = \frac{1}{2}\dot{\mathbf{x}}^{\boldsymbol{T}}\mathbf{D}(\mathbf{x})\dot{\mathbf{x}}, \tag{16}$$

where the mass matrix of the system is given by:

$$\mathbf{D}(\mathbf{x}) = \mathbf{J}_x^T(\mathbf{x})\mathbf{M}(\mathbf{x})\mathbf{J}_x(\mathbf{x}) + \mathbf{J}_{\varphi}^T(\mathbf{x})\mathbf{I}(\mathbf{x})\mathbf{J}_{\varphi}(\mathbf{x}). \tag{17}$$

4 VARIABLE MASS LAGRANGE APPROACH

In this section, the dynamics of cable-driven parallel manipulator is obtained by the variable mass Lagrange formulation. As the length of the cables in CDPM is a function of the moving-platform position, the cable mass changes in time. In fact, the mass that is added to or departed from the system will add or separate a momentum to the system. The Dynamics of mechanism with variable mass is discussed in detail by Cveticanin in [2], in which the Lagrange formulation is extended to:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right)^T - \left(\frac{\partial T}{\partial \mathbf{x}}\right)^T = \mathbf{q} + \mathbf{q}^{Fi} + \mathbf{d} + \mathbf{q}^{R^*}. \tag{18}$$

In this formulation, $\mathbf{q}$ and $\mathbf{q}^{Fi}$, are the generalized forces caused by non-conservative and conservative external forces acting on the system, respectively. Furthermore $\mathbf{d} + \mathbf{q}^{R^*}$ accounts for the effect of changing mass in the system. In other words, $\mathbf{q}^{R^*}$ is an impact force that is caused by the mass stream entering into or expelling from the system, and is a function of the mass variation and its relative velocity. Furthermore, $\mathbf{d}$ accounts for the direct energy that is added or removed to the system by entry or departure of the stream mass.

4.1 Kinetic Energy Terms

Let us examine the required terms of the Lagrangian formulation for CDPM. As usual the first two terms can be derived from the kinetic energy of the system given by (16):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right)^T - \left(\frac{\partial T}{\partial \mathbf{x}}\right)^T = \mathbf{D}(\mathbf{x})\ddot{\mathbf{x}} + \left(\dot{\mathbf{D}}(\mathbf{x}) - \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}(\dot{\mathbf{x}}^T\mathbf{D}(\mathbf{x}))\right)\dot{\mathbf{x}}, \tag{19}$$

where $\dot{\mathbf{D}}(\mathbf{x})$ is the time derivatives of the terms given by (17).

4.2 Generalized Forces

As explained in the extended Lagrange formula, $\mathbf{q}^{Fi}$ and $\mathbf{q}$ are respectively, the generalized forces caused by non-conservative, and conservative external forces acting on the system. The generalized force acting on the system caused by external non-conservative forces are composed of the following two elements:

$$\mathbf{q} = \mathbf{w}_x + \mathbf{q}_{nc}, \tag{20}$$

where $\mathbf{w}_x$ is the wrench (forces and torques) corresponding to the projection of the actuator forces (cable force) on the platform and $\mathbf{q}_{nc}$ is the external forces and torques acting directly on the moving-platform. According to the principle of virtual work and the Jacobians given by (3) and (6), the vector $\mathbf{w}_x$ can be obtained by projection of the actuator forces into the Cartesian space, using the manipulator Jacobian matrices as:

$$\mathbf{w}_x = [\mathbf{J}_{lx}(\mathbf{x}) \quad \mathbf{J}_{l\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x})]^{T}\ \boldsymbol{\tau} = \mathbf{J}_w^T(\mathbf{x})\boldsymbol{\tau}, \tag{21}$$

where $\boldsymbol{\tau}$ denotes the vector of the actuator forces (cable forces). The contribution due to the gravity forces may be expressed as the following potential energy:

$$V = \mathbf{g}^T\left(\mathbf{M}_1\mathbf{x}_p + \sum_{i=1}^{n} m_i(l_i)\mathbf{x}_{ci}\right), \tag{22}$$

where $\mathbf{g}$ is the gravity vector represented in the base frame and $\mathbf{x}_p$ is the position vector of the moving platform. According to [2], the potential energy can be expressed as function of the generalized coordinates. Therefore, $\mathbf{q}^{Fi}$ is obtained by partial derivative of the potential energy with respect to the generalized coordinates:

$$\mathbf{G}(\mathbf{x}) = -\frac{dV}{d\mathbf{x}}, \tag{23}$$

4.3 Variable Mass Terms

The formulation proposed for varying mass mechanism in [2] is defined for particle mass system. However, the additional terms caused by the variable mass are only function of mass derivatives (small variation of mass divided by small variation of time). For this reason and because these variations are continuous, the mass derivative acts as particle, even for body systems. This interpretation has already been considered in [2, 17] for the analysis of the vibration of varying mass mechanisms (see also [4]). As discussed in [2], the effect of changing mass in the system is caused by the variable momentum. This effect can be divided into the impact forces denoted by $\mathbf{q}^{R^*}$, and the energy that added or departed from the system by variable mass denoted by $\mathbf{d}$. Since cables are the only source of the variable mass and the variation is only function of the generalized coordinates, d_k can be determined by [17] :

$$d_k(\mathbf{x}, \dot{\mathbf{x}}) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial m_i(l_i)}{\partial x_k}\mathbf{v}_i^T\,\mathbf{v}_i, \tag{24}$$

where $\mathbf{v}_i$ is the velocity of the variable mass i and k denote individual generalized coordinates. According to figure 1, this mass variation is located at the beginning of the cable i and its velocity is in only one direction when it is expressed in the frame of the cable. For this reason, $\mathbf{v}_i$ can be considered as a scalar given by $\dot{l}_i$. Then, by using (2) and (11), (24) can be rewritten as

$$d_k(\mathbf{x}, \dot{\mathbf{x}}) = -\frac{1}{2}\,\rho_m\sum_{i=1}^{m}\frac{\partial f_{li}}{\partial x_k}\left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2. \tag{25}$$

On the other hand, the effect of the impact forces $q_k^{R^*}$ can be obtained from [17]:

$$q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \sum_{i=1}^{m} \dot{m}_i(l_i)\mathbf{v}_{oi}^T \frac{\partial \mathbf{p}_i}{\partial x_k}, \tag{26}$$

where $\mathbf{v}_{oi}$ is the velocity of the expelled or gained mass and $\mathbf{p}$ is the position of this mass variation. This variation is also located at the beginning of the cable i and its position variation as well as its velocity is in only one direction when they are expressed in the frame of the cable. For this reason, $\mathbf{v}_{oi}$ and the variation of $\mathbf{p}_i$ can be interpreted as scalars, respectively given by $\dot{l}_i$ and $\partial l_i/\partial x_k$. Then, by using (2) and (11), (26) can be rewritten as

$$q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \rho_m \sum_{i=1}^{m} \left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2 \frac{\partial f_{li}}{\partial x_k}. \tag{27}$$

Then, $d_k$and $q_k^{R^*}$ can be combined as

$$d_k(\mathbf{x},\dot{\mathbf{x}}) + q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}\rho_m \sum_{i=1}^{m} \frac{\partial f_{li}}{\partial x_k}\left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2 . \tag{28}$$

4.4 Final Dynamics Equations

From Equations (17), (18), (19), (21),(23) and (28), the general form of the dynamics of CDPM can be released in compact standard form as:

$$\mathbf{D}(\mathbf{x})\,\ddot{\mathbf{x}} + \mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{G}(\mathbf{x}) = \mathbf{J}_w(\mathbf{x})\boldsymbol{\tau} + \mathbf{q}_{nc}, \tag{29}$$

where $\mathbf{D}$ is given by (17), $\mathbf{G}$ is given by (23), $\mathbf{J}_w$ is defined by (21) and $\mathbf{c}$ is given by

$$\mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) = \left(\dot{\mathbf{D}}(\mathbf{x}) - \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}(\dot{\mathbf{x}}^T\mathbf{D}(\mathbf{x}))\right)\dot{\mathbf{x}} - \left(\mathbf{d}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{q}^{R^*}(\mathbf{x},\dot{\mathbf{x}})\right), \tag{30}$$

where each elements of $\mathbf{d} + \mathbf{q}^{R^*}$are given by (28). In (30), $\mathbf{D}$ is the mass matrix, $\mathbf{c}$ is the vector of centrifugal, Coriolis and mass variation terms and $\mathbf{G}$ is the vector of gravity terms. Finally $\mathbf{q}_{nc}$ is the external wrench vector acting directly on the moving-platform.

5 CASE STUDY

In this section, the dynamics of planar CDPM discussed in [12] (see figure 3) were considered. This

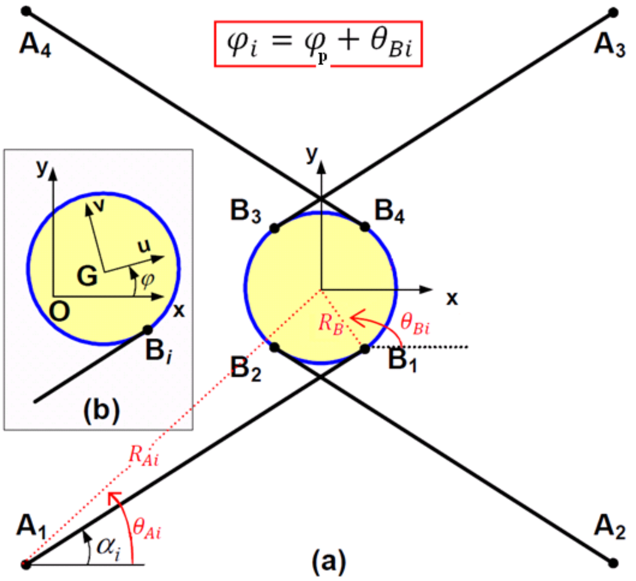


Figure 3. Simple schematic of planar CDPM.

CDPM is a simplified planar version adopted from the structure of Large Adaptive Reflector (LAR). This structure consists of a parallel redundant manipulators actuated by long cables. The control objective in the simplified mechanism is to track the position and the orientation of the moving platform as desired in presence of disturbance forces, such as wind turbulence. The geometric and inertial parameters used in the simulations of the system are adopted from LAR design. In this way, the length of the cables is in the order of 900 meters and the mass density of cables ρ_m is $0.215\ kg/m$. The main control purpose is the positioning of the moving-platform $\mathbf{x} = [x \quad y \quad \varphi_p]^{\boldsymbol{T}}$ which has the mass $M_p = 2500\ kg$. At first, the dynamics of the planar CDPM is obtained by the Lagrange method. Then, the effect of the variable mass in the cables is studied in detail.

From the inverse kinematics analysis the length of the cable l_i and the angle α_i can be obtained easily by writing the loop closure equations:

$$\begin{aligned} l_i &= [(x + R_B \cos\varphi_i - x_{Ai})^2 + (y + R_B \sin\varphi_i - y_{Ai})^2]^{\frac{1}{2}}, \\ \alpha_i &= atan2\big((y + R_B \sin\varphi_i - y_{Ai}), (x + R_B \cos\varphi_i - x_{Ai})\big). \end{aligned} \tag{31}$$

Also by Jacobians analysis we have:

$$\begin{aligned} \mathbf{J}_{xx} &= \begin{bmatrix} S_{1x} & S_{1y} \\ S_{2x} & S_{2y} \\ S_{3x} & S_{3y} \\ S_{4x} & S_{4y} \end{bmatrix}, \quad \mathbf{J}_{x\omega} = \begin{bmatrix} E_{1x}S_{1y} - E_{1y}\,S_{1x} \\ E_{2x}S_{2y} - E_{2y}\,S_{2x} \\ E_{3x}S_{3y} - E_{3y}\,S_{3x} \\ E_{4x}S_{4y} - E_{4y}\,S_{4x} \end{bmatrix}, \\ \mathbf{J}_{\omega x} &= \frac{1}{l_i}\begin{bmatrix} -S_{1y} & S_{1x} \\ -S_{2y} & S_{2x} \\ -S_{3y} & S_{3x} \\ -S_{4y} & S_{4x} \end{bmatrix}, \quad \mathbf{J}_{\boldsymbol{\omega\omega}} = \frac{1}{l_i}\begin{bmatrix} E_{1x}S_{1y} + E_{1y}\,S_{1x} \\ E_{2x}S_{2y} + E_{2y}\,S_{2x} \\ E_{3x}S_{3y} + E_{3y}\,S_{3x} \\ E_{4x}S_{4y} + E_{4y}\,S_{4x} \end{bmatrix}, \end{aligned} \tag{32}$$

where vectors $\mathbf{E}$ and $\hat{\mathbf{S}}$ are defined as:

$$\begin{aligned} [E_{ix} \quad E_{iy}]^T &= [R_B \cos(\varphi + \theta_{Bi}) \;\; R_B \sin(\varphi + \theta_{Bi})]^T, \\ \hat{S}_i = [S_{ix} \quad S_{iy}]^T &= [\cos\alpha_i \;\; \sin\alpha_i]^T. \end{aligned} \tag{33}$$

Moreover for planar CDPM, we have $\mathrm{J}_{\omega\varphi} = I$ and therefore, Jacobian matrices are easily defined by Equations (7) and (8). Finally, by deriving Equations (17), (30) and (23), the mass matrix $\mathbf{D}$, centrifugal, Coriolis and mass variation terms $\mathbf{c}$, and the vector of gravity terms $\mathbf{G}$ are obtained. Thus the dynamic modeling of planar CDPM is expressed as:

$$\mathbf{D}(\mathbf{x})_{3\times3}\,\ddot{\mathbf{x}}_{3\times1} + \mathbf{c}(\mathbf{x},\dot{\mathbf{x}})_{3\times1} + \mathbf{G}(\mathbf{x}) = [F_x \quad F_y \quad \tau_z]^T + \mathbf{q}_{nc}, \tag{34}$$

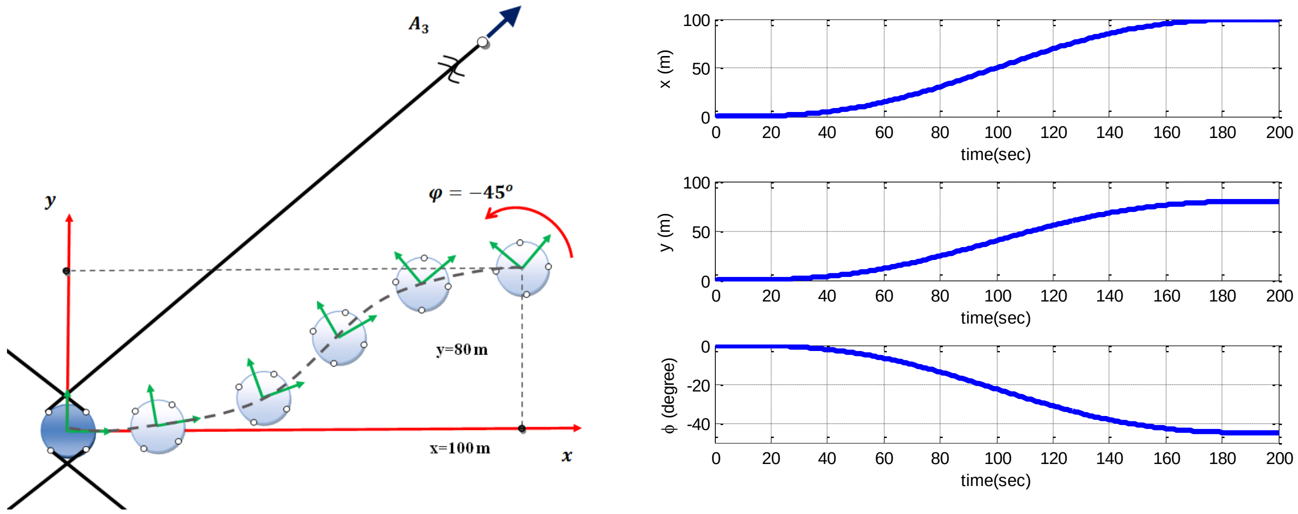


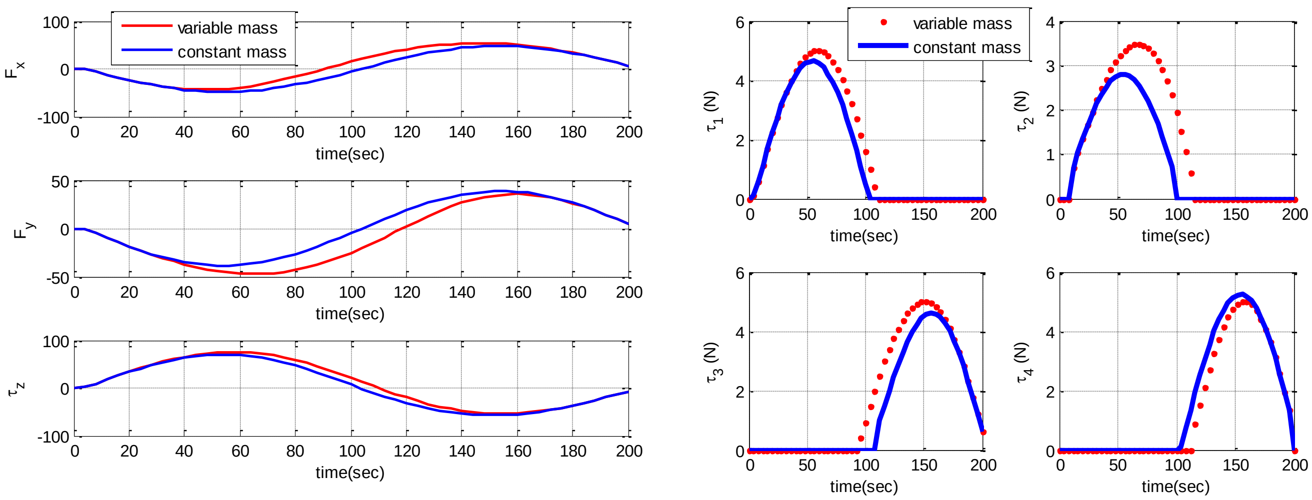
Figure 4. Desired trajectory.

where F_x, F_y and τ_z form the wrench applied on the moving platform, defined by:

$$[F_x \quad F_y \quad \tau_z]^T = \mathbf{J}_w(\mathbf{x})^T{}_{\mathbf{3\times4}}\boldsymbol{\tau}_{\mathbf{4\times1}}. \tag{35}$$

In (35), $\boldsymbol{\tau}_{\mathbf{4\times1}}$ is the vector of the forces in links space or, in other words, the tensions in the cables that are generated by the actuators (motors). As the Jacobian matrix in redundant manipulator is non-square, tension in the cables can be obtained by Redundancy Resolutions (optimal distribution of forces in cables) algorithms [[18], 19]. This resolution ensures positive tension in all cables.

For simulation, a specific displacement of the moving-platform is chosen. This simple trajectory is shown in figure 4. Then, the forces in Cartesian space are obtained by the inverse dynamic model given by (34). These forces are compared with the forces obtained by the same simulation, in which the effect of variable masses in the cables is neglected. Figure 5 (a) shows forces and torque in Cartesian space. Figure 5 (b) shows the projected forces in links space. In other words, it shows the tensions in the cables which are defined by Equation (35) as $\boldsymbol{\tau}_{\mathbf{4\times1}} = [\tau_1, \tau_2, \tau_3, \tau_4]^T$. These forces were obtained by driving the numerical algorithm used to solve the "non-negative least-squares constraints problem" described in [19] and implemented in the Matlab optimization toolbox. As we expect from the dynamics equation analysis, the variable mass have an important effect in the dynamics of the manipulator. In application such as LAR project [12], the length and mass density of the cables are important. In this context, the variable mass of the cables plays a vital role in the dynamics of CDPM. Moreover, the effects of the variable mass in the cables are strongly dependent to the position and the velocity trajectories. This effect is non-linear and is dependent to the parameters such as the cables mass density, the mass of the moving-platform, and the kinematics structure. In fact, the additional effect of the variable mass is completely described by Equation (28). Therefore, this effect is directly proportional to the cable mass density ρ_m. This parameter could reduce the effect of the variable mass. However, this reduction would increase the flexibility of the cables, which is not necessarily a better thing. In addition, since $\mathbf{f}_l(\mathbf{x})$ in Equation (2) is a kinematic function of the moving-platform position, the variable mass effect is strongly dependent to the size and the topology of the CDPM.



(a) (b)

Figure 5 (a) Forces and torque in Cartesian space (Moving-platform workspace).

(b) Tensions in cables (Forces in joint space).

6 CONCLUSIONS

This paper focused on the dynamics modeling of Cable-Driven Parallel Manipulators (CDPMs) by using Lagrange formulation. While in previous works the effect of entering mass stream to the system caused by elongation of the cables are neglected, in this paper, this effect is treated by using a Lagrange variable mass formulation. By this means, a complete dynamics of the system is derived, while the compact and tractable closed-form dynamics formulation is preserved. In this way, at first a general formulation for a general CDPM is given, where the effect of mass variation in the cables is integrated into its dynamics. The significance of such treatment is appreciated in a complete analysis of the dynamics, vibrations, stability of such systems, and in any robust control synthesis of such manipulators. The general formulation is applied to a typical planar CDPM with cables of 900 meters length. Through simulations, the validity and integrity of the obtained formulation are firstly verified, and then, significance of variable mass treatment in such analysis is examined. It is shown that the effect of entering mass stream into the system is not negligible, while it is non-linear and strongly dependent to the geometric and mass parameters of the robot, as well as the maneuvering trajectory.

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