CONVERGENCE TIME ESTIMATION FOR JUMPING MECHANICAL SYSTEMS USING SECOND ORDER SLIDING MODES

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ABSTRACT

In this paper, a family of second order sliding mode controllers affected with uniformly bounded disturbances and rigid body inelastic impacts is presented. The considered systems are modeled by a double integrator affected with jumps in velocity and bounded external perturbations. It is shown that using a non-smooth strict Lyapunov function and a non-smooth coordinate transformation it is possible to ensure finite time stability and the convergence time of the closed loop system without having to analyze the Lyapunov function at the jump instants. The performance and robustness properties of the feedback synthesis are illustrated with numerical experiments using a one-link pendulum with physical constraints as a testbed.

Keywords: Second-order sliding modes; Lyapunov function; Stability analysis.

SUR LA CONVERGENCE EN TEMPS FINI POUR SYSTÈMES MÉCANIQUES IMPULSIFS À BASE DE MODES GLISSANTS DU SECOND ORDRE

RÉSUMÉ

Cet article est consacré à la synthèse d'une famille de contrôleurs par mode glissant de second ordre affectés par des perturbations qui sont uniformément bornées et des chocs inélastiques entre corps rigides. Les systèmes considérés sont modélisés par un double intégrateur aux des sauts de vitesse et des perturbations externes bornées. Il est montré qu'en utilisant une fonction de Lyapunov stricte non-régulière et une transformation des coordonnées non-régulière, il est possible d'assurer la stabilité en temps fini et le temps de convergence du système en boucle fermée sans analyser la fonction de Lyapunov aux instants de saut. Des résultats de simulation sont également donnés qui illustrent, d'autre part, les propriétés de performance et de robustesse d'une telle technique en utilisant un pendule à un lien avec des contraintes physiques comme plateforme d'essai.

Mots-clés : Mode Glissant du Second Ordre ; Fonction de Lyapunov ; Analyse de la stabilité.

1. INTRODUCTION

The analysis and the implementation of a closed loop mechanical system is always affected by nonlinear perturbations such as friction, growing uncertainties or physical constraints. Sliding mode algorithms (SOSM) are a popular option because of their properties such as finite time convergence to the origin in spite of the presence of bounded, persisting external disturbances and parametric uncertainties. In this paper, a generalization of the twisting algorithm is studied in order to design continuous finite time-stabilizing feedback controllers (see [1, 2]).

A linear double integrator, *i.e.* a mechanical system with Coulomb friction and uniformly growing perturbations, is considered with jumps in its velocity when it hits a constraint surface. The velocity undergoes an instantaneous jump when the inelastic collision occurs. In this paper, the restitution in velocity, representing loss of energy which occurs at the time of impact, is considered fully known. It is clear from the literature (for example see [3–5]) that a jump in the Lyapunov function occurs whenever the position satisfies the constraint of the restitution rule and the velocity undergoes a discontinuity. Therefore the Lyapunov stability needs to be specifically defined for all possible jumps in the Lyapunov function. The rigorous theoretical developments in the theory of non-smooth mechanics have been accompanied by applications such as biped robots (see for example [6–8]).

This paper is based on the ideas of [9] and [10], a mechanical system with resets in velocity, affected by bounded external perturbations with two main objectives. First, a non-smooth state transformation [4] is utilized to render a jump-free system with its solutions clearly defined in the sense of Filippov [11]. Second, as in [12], the stability analysis of the closed loop system is made within non-smooth strict Lyapunov methodology for discontinuous systems. There is well-known theory that study these concepts, for example [13–16]. Indeed, in [9] homogeneity properties are used to analyze the stability of the system and estimate an upper bound for convergence time of the closed loop system. Another work that uses homogeneity properties of a discontinuous system is [17].

Moreover, based on previews work, such as [12], a non-smooth strict Lyapunov function is identified to show global finite time stability of the closed loop system using a family of controllers without having to analyze the jumps in the Lyapunov function. Finally, the sliding mode synthesis is shown to stabilize the double integrator with impacts in finite time, in spite of uniformly growing perturbations. Moreover, an estimation of the convergence time of the trajectories of the closed loop systems is obtained.

The main contribution of this paper two-fold: (a) to provide a sketch of the stability analysis of the closed-loop system, and (b) to illustrate using numerical experiments the performance of the control design for the disturbed and disturbance-free scenarios. A continuous stabilizing feedback controller is implemented on a one-link pendulum, affected by Coulomb friction, growing perturbations with respect to the state and velocity jumps, as a test bed.

The structure of the paper is as follows: basic assumptions of the systems under interest and some mathematical background are given in Section II. In Section III, a sketch of the stability of the unperturbed and the system affected by bounded perturbations are analyzed. In both cases, finite time stability for the origin of the closed loop system is concluded. In order to support the theoretical results, the proposed control law is implemented in Section IV, on a one-link pendulum with physical constraints as a test bed. Finally, Section V presents the conclusions of this work.

2. PROBLEM STATEMENT

The general model of second-order mechanical systems, written in the state space form is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= f(x,y) + \tau + \delta(t,x,y), \\ F_0(x,t_k) &\geq 0, \\ y(t_k^+) &= -\bar{e}y(t_k^-) \; if \; F_0(x,t_k^-) = 0, \end{aligned} \tag{1}$$

where x and y are the position and the velocity respectively, τ is the control input, and the nominal known part of the system dynamics is represented by the function $f(x,y)$, while the uncertainties are concentrated in $\delta(t,x,y)$. The first two lines of (1) occur beyond the constraint surface $F_0(x,t_k) \geq 0$, often referred to as unilateral constraint. The k^{th} jump time instant, where the velocity undergoes a reset or jump, is given by t_k, $\bar{e}$ denotes the loss of energy and $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits respectively of y at the jump time t_k. The third equation represents the dynamics with unilateral constraints on position x (see for example [9]). It is assumed that the jump event occurs instantaneously within an infinitesimally small time, when the constraint $F_0(x,t_k) = 0$ is met, and hence mathematically can be represented by *Newton's restitution rule* given by the fourth equality of Eq. (1).

For system (1) the following controller design is proposed

$$\tau = U - f(x,y), \tag{2}$$

where U is a new control input.

$$U = -k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^{\alpha}\operatorname{sgn}(y), \tag{3}$$

where k_1, k_2 are positive constants and $0 \leq \alpha \leq 1$.

Let consider an external bounded perturbation $\delta(x,y)$ given by

$$|\delta(x,y)| \leq \mu_y|y|^{\alpha} + \mu_x|x|^{\frac{\alpha}{2-\alpha}}, \tag{4}$$

where μ_y, $\mu_x \in I\!R$ are positive constants and α is defined as Eq. (3). Therefore, system (1) is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= U + \delta(x,y). \end{aligned} \tag{5}$$

Note that the analyzed controllers in Eq. (3) are a continuous family of control algorithms that contain from twisting algorithm to PD control law acting in the closed-loop system (5). Indeed, if $\alpha = 0$, the algorithm (3) is known as twisting algorithm, and if $\alpha = 1$ it is not difficult to see that the algorithm (3) is known as PD control law.

External disturbances, vanishing in the origin and satisfying the growth condition (4) are considered in all cases but $\alpha = 0$. Indeed, according to [[17], Theorem 4.2], the disturbed system (5) renders the finite time stability, regardless of whichever disturbance δ with a uniform upper bound

$$|\delta(x,y)| \leq M, \tag{6}$$

for $t \geq 0$, affects the system provided that

$$0 < M < k_2 < k_1 - M. \tag{7}$$

A non-smooth transformation is employed to transform the impact system (5) into a jump free system (see [5], chapter 2). Let the non-smooth transformation be defined as follows:

$$x = |s|, y = R\nu\mathrm{sgn}(s), R = 1 - k\mathrm{sgn}(s\nu), k = \frac{1-\bar{e}}{1+\bar{e}}. \tag{8}$$

The variable structure-wise transformed system

$$\begin{aligned} \dot{s} &= R\nu, \\ \dot{\nu} &= R^{-1}\mathrm{sgn}(s)\left(U(|s|, R\nu\mathrm{sgn}(s)) + \delta(|s|, R\nu\mathrm{sgn}(s))\right), \end{aligned} \tag{9}$$

is then derived by employing (8) (see [5]). Considering again the non-smooth transformation, the uncertainty term is denoted by

$$\delta(|s|, R\nu\mathrm{sgn}(s)) = \mu_s|s|^{\frac{\alpha}{2-\alpha}} + \mu_\nu|R\nu|^{\alpha}. \tag{10}$$

By combining (8), the controller (3) can be represented in the transformed coordinates as follows:

$$U(|s|, R\nu\mathrm{sgn}(s)) = -k_1|s|^{\frac{\alpha}{2-\alpha}} - k_2|R\nu|^{\alpha}\mathrm{sgn}(R\nu s). \tag{11}$$

Substituting Eq. (11) into Eq. (9), the closed-loop system in the new coordinate frame can be obtained as follows:

$$\begin{aligned} \dot{s} &= R\nu, \\ \dot{\nu} &= -k_1R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\mathrm{sgn}(s) - k_2R^{\alpha-1}|\nu|^{\alpha}\mathrm{sgn}(\nu) + R^{-1}\delta(|s|, R\nu\mathrm{sgn}(s))\mathrm{sgn}(s). \end{aligned} \tag{12}$$

The origin $s = \nu = 0$ of the unperturbed system (12) corresponds to the origin $x = y = 0$ of the unperturbed system (5). Note that the transformation is not invertible, one starts from the closed-loop system (12) and the original dynamics can be recovered via Eq. (8). The solutions of system (12) are well defined in the sense of Filippov. Furthermore, such formulation admits both friction and jump phenomena, while guaranteeing existence of solutions.

Previous results for the considered systems, defined above, are shown in the next section.

2.1. Mathematical Background

Some contributions, fundamental for the rest of this work, are now recalled. The notation of some theorems was modified for readability.

The stability analysis of the system

$$\begin{aligned} \dot{s} &= R\nu, \\ \dot{\nu} &= -R^{-1}\Big(\alpha\mathrm{sgn}(s) + \beta\mathrm{sgn}(\nu) - \mathrm{sgn}(s)\delta(t)\Big), \end{aligned} \tag{13}$$

was considered in [10], where a strict Lyapunov function was proposed in order to show finite time stability of the system (13).

Theorem 1 *[10] System (13) has finite time convergence to the point $(x, y) = (0, 0)$ if*

$$\alpha - M > \beta > M, \alpha > \left(\frac{\gamma_2}{2\gamma_1}R\right)^{\frac{2}{3}},\ \alpha\Big(\beta - M\Big)^2 > \frac{9}{32}\left(\frac{\gamma_2}{\gamma_1}R\right)^2, \tag{14}$$

holds, with

$$t_{reach} \leq \frac{4}{\zeta}\eta^{\frac{3}{4}}V^{\frac{1}{4}}(x(0), y(0)), \tag{15}$$

as an estimation of the convergence time, with

$$\zeta \quad = \quad min\left\{\frac{3}{2}\gamma_2 R, \gamma_2 R^{-1}\Big(\alpha-\beta-M\Big), 2\gamma_1\alpha(\beta-M)R^{-1}, \gamma_1(\beta-M)R\right\}, \tag{16}$$

and

$$\eta = max\left\{\gamma_1\alpha^2 R^{-2}, \gamma_2, \gamma_1\alpha, \frac{1}{4}\gamma_1 R^2\right\}. \tag{17}$$

The disturbed double integrator, described by equations of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^{\alpha}\operatorname{sgn}(y) + \delta(x,y,t), \end{aligned} \tag{18}$$

was under study in [12].

Theorem 2 *[12] System (18) has finite-time convergence to the point $(x,y) = (0,0)$ with*

$$t_{reach} \leq \frac{1}{\zeta_{minp}}\gamma_{max}^{\frac{3+\alpha}{4}} V^{\frac{1-\alpha}{4}}(x(0), y(0)), \tag{19}$$

as an estimation of the convergence time of the trajectories of the closed loop system (5), with

$$\begin{aligned} \zeta_{minp} &= min\left\{\eta_x\eta_y - \frac{3}{2}\frac{1+\alpha}{2-\alpha}, \frac{1}{2-\alpha}\left(\eta_y - \frac{3}{2}(1-\alpha)\right), \eta_x - \eta_y\frac{1}{1+\alpha},\ \eta_x - \frac{\alpha}{1+\alpha}\right\}, \\ \gamma_{max} &= max\left\{\lambda_{max}(P_2), \frac{1}{2(2-\alpha)}\right\}, \end{aligned} \tag{20}$$

and

$$P_2 \quad = \quad \begin{pmatrix} \frac{2-\alpha}{2}(k_1-\mu_x sgn(x))^2 & \frac{1}{2} \\ \frac{1}{2} & \frac{2-\alpha}{2}(k_1-\mu_x sgn(x)) \end{pmatrix}, \tag{21}$$

$$\begin{aligned} k_1 &> \mu_x + \frac{1}{1+\alpha}max\left\{\eta_y,\ \alpha,\ \frac{1+\alpha}{(2-\alpha)^{\frac{2}{3}}}\right\}, k_2 > \mu_y + \frac{3}{2}(1-\alpha), \\ \eta_x\eta_y &> \frac{3}{2}\left(\frac{1+\alpha}{2-\alpha}\right), \eta_x = k_1-\mu_x,\ \eta_y = k_2-\mu_y, \end{aligned}$$

for $0 \leq \alpha < 1$ and asymptotic stability for $\alpha = 1$.

The main results of this paper will be developed in the following sections.

3. MAIN RESULTS

3.1. The Unperturbed System

Consider the unperturbed system (12) given by

$$\begin{aligned} \dot{s} &= R\nu, \\ \dot{\nu} &= -k_1 R^{-1}|s|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(s) - k_2 R^{\alpha-1}|\nu|^{\alpha}\operatorname{sgn}(\nu), \end{aligned} \tag{22}$$

where k_1, k_2 are positive constants and $0 < \alpha < 1$.

Theorem 3 *Let*

$$V(s,\nu) = \frac{2-\alpha}{2}R^{-2}k_1^2|s|^{\frac{4}{2-\alpha}} + k_1|s|^{\frac{2}{2-\alpha}}\nu^2 + \gamma|s|^{\frac{3}{2-\alpha}}sgn(s)\nu + \frac{1}{2(2-\alpha)}R^2\nu^4 \tag{23}$$

a strict nonsmooth Lyapunov function for the system (22) *, with* $\gamma > 0$*. Then, the system* (22) *has finite time convergence to the point* $(s,\nu) = (0,0)$ *if*

$$\begin{aligned} k_1 &> max\left\{\frac{1}{2(2-\alpha)}\gamma(R)^{\frac{2}{3}}, \frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}R^{\alpha}k_2\right\}, \textit{ and} \\ k_2 &> \frac{3}{4}(1-\alpha)\gamma R^{-\alpha};\ k_1k_2 > \left(\frac{3}{2}\right)\frac{1+\alpha}{2-\alpha}\gamma R^{2-\alpha} \end{aligned} \tag{24}$$

holds, with

$$t_{reach} \le \zeta^{-1}\eta^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(s(0),\nu(0)), \tag{25}$$

as an estimation of the convergence time, with

$$\begin{aligned} \zeta = min\Bigg\{ & R^{\alpha-1}k_2\left(k_1 - \frac{\alpha}{1+\alpha}\right), R^{-1}k_1 - \frac{1}{1+\alpha}R^{\alpha-1}k_2, \\ & R^{\alpha-1}k_1k_2 - \frac{3(1+\alpha)}{2(2-\alpha)}R;\ \frac{2}{2-\alpha}R^{\alpha}k_2 - \frac{3}{4}\cdot\frac{1-\alpha}{2-\alpha}\Bigg\}, \end{aligned} \tag{26}$$

and

$$\eta = max\left\{\lambda_{\max}(P_3), \frac{1}{2(2-\alpha)}R^2\right\}, \tag{27}$$

with

$$P_3 = \begin{pmatrix} \frac{2-\alpha}{2}R^{-2}k_1^2 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & k_1 \end{pmatrix}. \tag{28}$$

Sketch of the proof: The jump-free non-smooth transformed system (22) is analyzed using the non-smooth strict Lyapunov function (23). Finite time convergence of the trajectories of the closed loop system (22) is shown by demonstrating that the time derivative of the Lyapunov function (23) is negative definite using the terms (24)-(28). Moreover, it is possible to rewrite the time derivative of the non-smooth Lyapunov function in terms of the same Lyapunov function, and by means of a comparison system, obtain an estimation of the time convergence of the trajectories to the point the point $(s,\nu) = (0,0)$. It is easy to show that if (s,ν) goes to zero then the state variable (x,y) goes to zero.

In the next section, the stability analysis of the perturbed system will be treated considering external bounded perturbation.

3.2. The Perturbed System

In this section a new strict non-smooth Lyapunov function is proposed to show finite time convergence of the trajectories of the perturbed system (12) to the point $(s,\nu)=(0,0)$ in spite of external perturbations bounded by inequality (10).

Theorem 4 *Let*

$$V(s,\nu)=\frac{2-\alpha}{2}\left(k_1-\mu_s sgn(s)\right)^2 R^{-2}|s|^{\frac{4}{2-\alpha}}+\left(k_1-\mu_s sgn(s)\right)|s|^{\frac{2}{2-\alpha}}\nu^2+\gamma|s|^{\frac{3}{2-\alpha}}sgn(s)\nu+\frac{1}{2(2-\alpha)}R^2\nu^4. \tag{29}$$

a strict nonsmooth Lyapunov function of the system (12)*. Then, the system* (12) *has finite time convergence to the point* $(s,\nu)=(0,0)$ *with*

$$t_{reach}\leq\frac{1}{\zeta_p}\psi^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(s(0),\nu(0)), \tag{30}$$

as an estimation of the convergence time of the trajectories of the closed loop system (5)*, with*

$$\begin{aligned}\zeta_p &= min\left\{\eta_s\eta_\nu R^{\alpha-1}-\frac{3}{2}\frac{1+\alpha}{2-\alpha}R\gamma,\frac{1}{2-\alpha}R\left(2\eta_\nu R^\alpha-\frac{3}{2}(1-\alpha)\gamma\right),\right.\\ &\left. R^{-1}\gamma\left(\eta_s-\frac{1}{1+\alpha}\eta_\nu R^\alpha\right),\eta_\nu R^{\alpha-1}\left(\eta_s-\frac{\alpha}{1+\alpha}\gamma\right)\right\}, \qquad (31)\\ \psi &= max\left\{\lambda_{\max}(P_4),\frac{1}{2(2-\alpha)}\right\},\end{aligned}$$

with

$$P_4=\begin{pmatrix}\frac{2-\alpha}{2}\left(k_1-\mu_s sgn(s)\right)^2R^{-2} & \frac{1}{2}\gamma\\ \frac{1}{2}\gamma & \left(k_1-\mu_s sgn(s)\right)\end{pmatrix}, \tag{32}$$

$$\begin{aligned}k_1 &> \mu_s+\frac{1}{1+\alpha}max\left\{\eta_\nu R^\alpha,\ \alpha\gamma,\ \left(\frac{1}{(2-\alpha)}\gamma\right)^{\frac{2}{3}}(1+\alpha)\right\},\\ k_2 &> \mu_\nu+\frac{3}{4}(1-\alpha)\gamma R^{-\alpha},\ \eta_s\eta_\nu>\frac{3}{2}\left(\frac{1+\alpha}{2-\alpha}\gamma R^{2-\alpha}\right), \qquad (33)\\ \eta_s &= k_1-\mu_s,\ \eta_\nu=k_2-\mu_\nu,\end{aligned}$$

for $0\leq\alpha<1$ *and asymptotic stability for* $\alpha=1$.

Sketch of the proof: The proof is very similar to the disturbance-free scenario. The disturbed jump-free non-smooth dynamics (12) is analyzed using a different non-smooth strict Lyapunov function (29). Finite time convergence of the trajectories of the closed loop system (12) is shown by demonstrating that the time derivative of the Lyapunov function (29) is negative definite using the terms (31)-(33). Once again, it is possible to rewrite the time derivative of the non-smooth Lyapunov function in terms of the same Lyapunov function, and by means of a comparison system, obtain an estimation of the time convergence of the trajectories to the point the point $(s,\nu)=(0,0)$, in spite of the bounded perturbations.

In the next section, in order to support theoretical results a numerical experiment is under study using a one link pendulum as testbed.

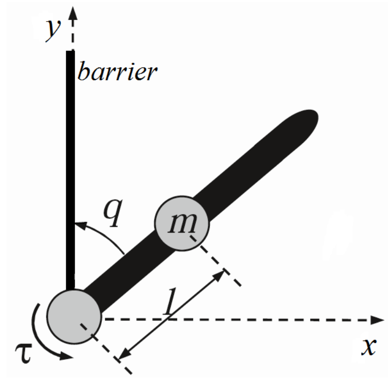


Fig. 1. Pendulum-barrier system.

4. NUMERICAL EXPERIMENTS

A simple testbed of an impacting pendulum is depicted in Fig. 1, where the free motion of the pendulum is confined by the barrier located at the positive y axis. For the free-motion dynamics $q \in (0, 2\pi)$, the plant equation reads

$$(ml^2 + J)\ddot{q} = -mgl\sin(q) - k\dot{q} + \tau + w_1, \tag{34}$$

where q is the angle made by the pendulum with the vertical, m is the mass of the pendulum, l is the distance to the center of mass, J is the moment of inertia of the pendulum about the center of mass, g is the gravity acceleration which acts in the vertical direction, k is a viscous friction coefficient, τ is the control torque, and w_1 stands for non-modeled external force such as dry friction. For the transition phase at the unstable equilibrium $q = 0$, the restitution rule is given by

$$q^+ = q^-, \quad \dot{q}^+ = -e\dot{q}^-, \quad e \in [0, 1]. \tag{35}$$

A similar velocity restitution occurs at $q = 2\pi$ however for certainty the subsequent local synthesis is confined to the free-motion domain $q \in (0, \pi)$ within the right half-plane.

To address the state feedback tracking of a reference trajectory with impacts $q^r(t)$, the state error variables

$$x_1 = q - q^r, \; x_2 = \dot{q} - \dot{q}^r, \tag{36}$$

are involved. A pre-feedback control law in the form

$$\tau = mgl\sin(q) + k\dot{q} + (ml^2 + J)(\ddot{q}^r + u), \tag{37}$$

is proposed, composed of a controller u to be designed and the rest being a feedback linearizer. Then, setting $\mathbf{x} = (x_1, x_2)^\top$, and rewriting the system (34)-(37) in terms of the tracking error variables, one derives the free-motion phase error dynamics

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u + w_1, \end{aligned} \tag{38}$$

within the constraint $F_0(\mathbf{x}, t) = x_1 + q^r(t) > 0$, and the transition phase error system

$$\begin{aligned} x_1^+ &= x_1^-, \\ x_2^+ &= \mu_0(\mathbf{x}, t), \end{aligned} \tag{39}$$

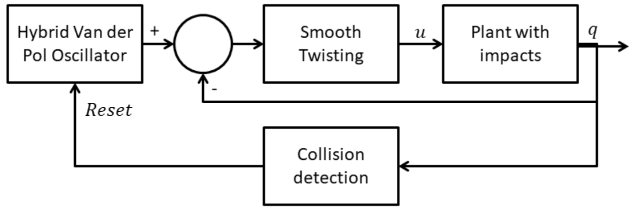


Fig. 2. Block-diagram of online Van der Pol reference model reset.

on the constraint surface $F_0(\mathbf{x},t) = x_1 + q^r(t) = 0$ where

$$\mu_0(\mathbf{x},t) = \begin{cases} \mu_1(\mathbf{x},t) & \text{if } F_0(q^r(t)) = 0,\ \ F_0(x_1+q^r) \neq 0 \\ \mu_2(\mathbf{x},t) & \text{if } F_0(q^r(t)) \neq 0,\ \ F_0(x_1+q^r) = 0 \\ \mu_3(\mathbf{x},t) & \text{if } F_0(q^r(t)) = 0,\ \ F_0(x_1+q^r) = 0, \end{cases} \tag{40}$$

with

$$\mu_1(\mathbf{x},t) = x_2 + (1+e)\dot{q}^r \tag{41}$$

$$\mu_2(\mathbf{x},t) = -e(x_2+\dot{q}^r) - \dot{q}^r, \tag{42}$$

$$\mu_3(\mathbf{x},t) = -ex_2 \tag{43}$$

corresponding to the three different possible scenarios that might occur: (1) the reference system trajectory hits the barrier before the pendulum; (2) the pendulum hits the barrier before the reference trajectory; (3) synchronized impact of the pendulum and the reference trajectory.

The reference trajectory is given by the hybrid Van der Pol Oscillator [18], which under the appropriate parameter selection, exhibits an asymptotically stable limit cycle with jumps. The reference dynamics between impacts, are given by

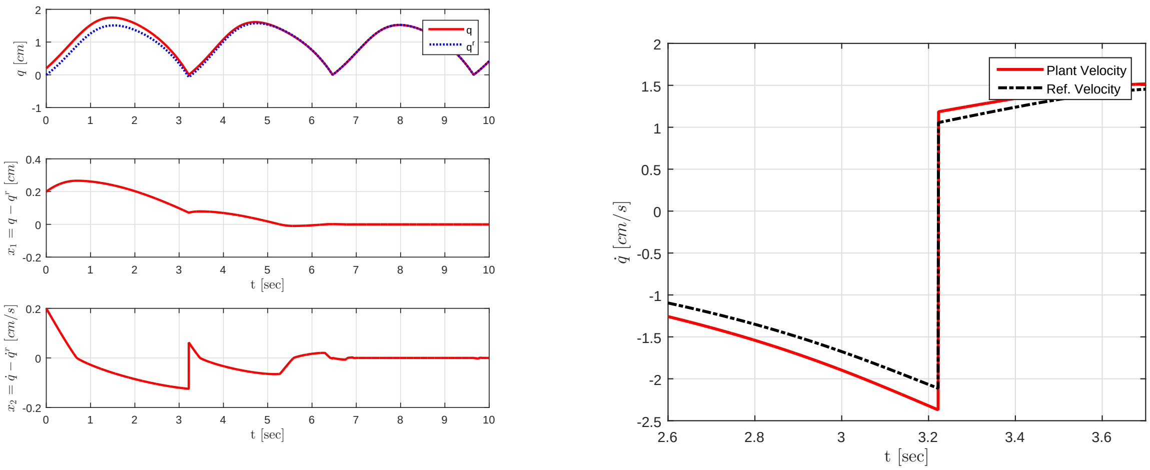
$$\ddot{q}^r = \mu(k - q^{r2})\dot{q}^r - q^r. \tag{44}$$

In this work, to suppress the peaking phenomena, characteristic of systems with jumps, which destroys the asymptotic stability of the disturbance-free closed loop system [19], the reference model is now reset online, as it is shown in the block-diagram of Fig. 2. Following the trajectory adaptation method presented in [20] The idea behind such a reset is in using the same hybrid Van der Pol reference model, but instead of using its own unilateral constraint $q^r = 0$, the reset event is synchronized with the impact of the plant ($q = 0$), so as to generate an asymptotically-stable limit cycle on the resulting full order dynamics. Thus, the restitution law of the hybrid Van der Pol oscillator is set to

$$q^r(t_i^+) = 0, \quad \dot{q}^r(t_i^+) = -e\dot{q}^r(t_i^-), \quad \text{if and only if } q(t_i) = 0. \tag{45}$$

The pre-feedback controller and controller u (37), are now coupled to the Van der Pol reference model, thus modified. Since the reference trajectory is reset when the plant hits the constraint, (43) is now in order, so the error in the transition phase is governed by $x_2^+ = -ex_2^-$. This in turn, provides validity to the non-smooth transformation (13), and the control action (3) can be implemented to the system above, by setting $x = x_1$, and $y = x_2$.

The numerical experiments below were performed using the following system values: $m = 1$, $k = 1$, $l = 1$, $g = 9.81$, $J = 1$, and $e = 0.5$. The hybrid Van der Pol oscillator values used were $\mu = 1$, $k = 1$, and $e = 0.5$. The simulations were done in Matlab/Simulink, using a Runge-Kutta solving method, with $1ms$ time step.



(a) Tracking Errors

(b) Impact Synchronization

Fig. 3. (a) Plots of the position, position error, and velocity error, and (b) plant and reference velocity, around the first impact, in the disturbance-free case when the online reset adaptation of the Van der Pol reference model is enforced.

First, an experiment with a slow convergence controller was performed by using the controller values $k_1 = k_2 = 0.2$, $\alpha = 0.2$. No disturbance was added to the system, *i.e.* $w_1 = 0$. The tracking errors are presented in Fig. 3(a). There we can find that the velocity tracking error presents jumps ($3.2s$), whereas the position tracking error is continuous, and both of them decrease to zero.

The plant jump and the reference are synchronized, as depicted in Fig. 3(b), and the error between them is clearly reduced after the jump. The pendulum trajectory thus slowly converges to the limit cycle of the hybrid Van der Pol Oscillator. This is illustrated in Fig. 4.

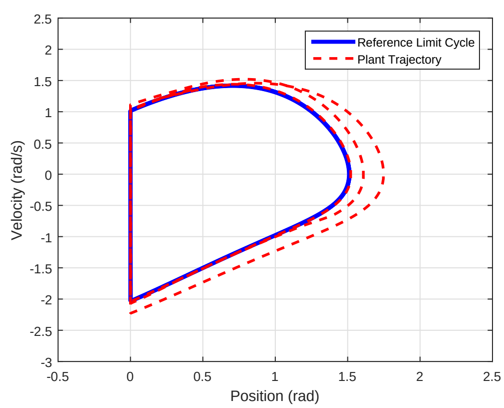


Fig. 4. Plots of the phase diagrams of the Pendulum system and the hybrid Van der Pol Oscillators.

A comparison between the dynamics of the Pendulum velocity error dynamics, and the evolution of non-smooth v in the transformed system is presented in Fig. 5, to illustrate that whereas the original system exhibits discontinuities, the transformed dynamics does not, and decreases to zero as a consequence of the main Theorem presented in Section 3.

Finally, a disturbance of $w_1 = 0.1\dot{q} + 0.1sign(\dot{q})$, accounting for viscous and dry friction, was added to

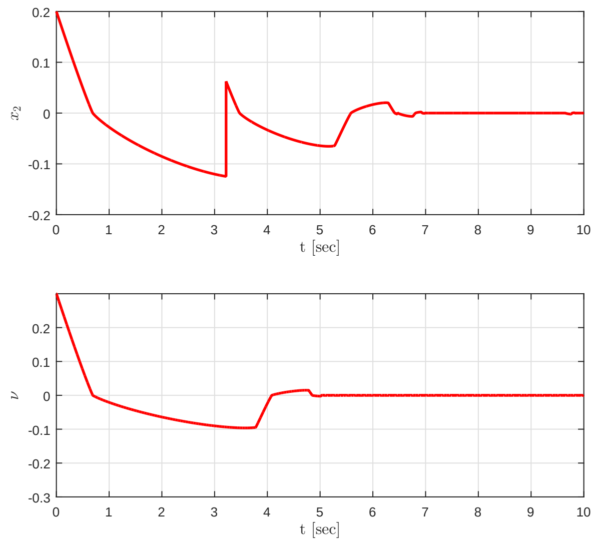


Fig. 5. Plots of the pendulum velocity error and μ of the non-smooth transformation dynamics.

the Pendulum dynamics, to show the performance of the control action u. The controller gains are set to $k_1 = k_2 = 10$, whereas the performance of the controller is compared for $\alpha = 0.2$ and $\alpha = 0.6$. In both cases, the error dynamics converge to zero, as shown in Fig. 6(a).

As introduced in Section 2, (3) degenerates to a twisting algorithm when $\alpha = 0$, and to a PD controller when $\alpha = 1$. Therefore, it is to be expected that lower values of α will produce a bigger chattering than higher values. This is indeed the case, as it is illustrated in Fig. 6(b).

5. CONCLUSIONS

A generalization of a second-order sliding mode controller "Twisting" is tuned such as global finite time exact convergence with respect to the growing perturbations is shown. With this aim a non-smooth strict Lyapunov function is proposed allowing an estimation of the upper bound of the convergence time. The performance of the proposed algorithm was shown by solving the tracking control problem of a one-link pendulum in spite of bounded external and parametric perturbations. The closed loop mechanical system showed to be robust and provide nice performance in spite of unknown but bounded uncertainties. For future work, this result can be easily generalized for multidimensional case. Moreover, it can be extended when a state variable is not available for measurement, then a finite time observer can be applied such as super-twisting algorithm.

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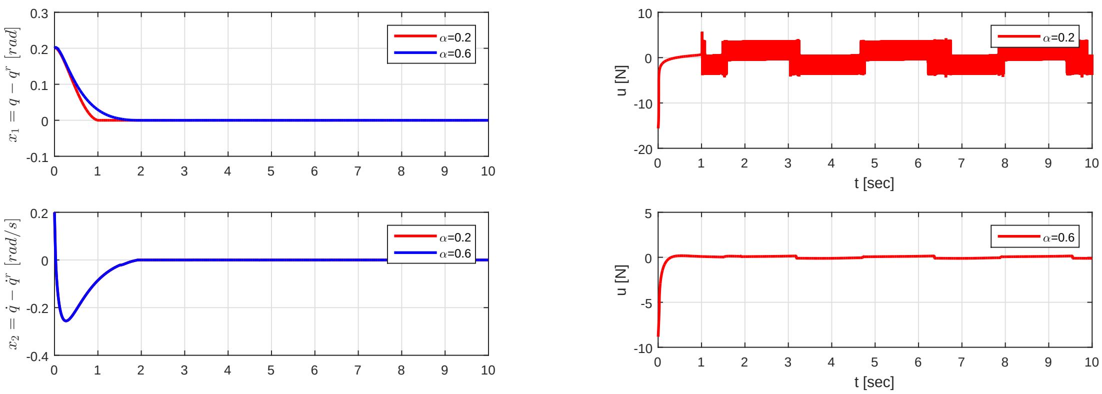

(a) Tracking Errors (b) Control Action

Fig. 6. Plots of the tracking errors and control action for $\alpha = 0.2$, and $\alpha = 0.6$, with $k_1 = k_2 = 10$.

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